Statistical Inference for Some Excursion Characteristics of Gaussian Process

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الأهـــــــــــــــداء

إلى الروح الطاهرة التي اضاءت لي الطريق، فاللهم أرحم والدي واغفر له، وأرض عنه رضواناً تحل به عليه جوامع رضوانك، وتحله به داركرامتك،،،

والـدي...

إلى من ملأت صدرها لي حناناً، وجوارحها لطفاً، وقلبها حباً، ودربي نوراً، وحثتني على طلب العلم والاجتهاد،،،

أمـي...

إلى من صبروا اثناء دراستي، وكانواخيرعون لي

زوجتي وأولادي (شيماء، زياد، حسين)،،،

إلى من وقفوا الى جانبي وحثوني على طلب العلم

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لكم جميعاً إهدي هذا العمل أحتراماً وعرفاناً

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LIST OF ABBREVIATIONS

u	Threshold
T	Time
(I_{k+1}, n_{k+1})	Represent partial intervals
S_N	Represent the durations of $X(t)$
λ, σ	Parameters of distribution of duration
δ	Length of duration
k	Number of intervals
α	Expected number of crossings
$\hat{\lambda}_{MLE}$	Maximum likelihood estimator for λ
$\hat{\lambda}_{B,1}$	Bayes estimator of λ
$\hat{\lambda}_{GMLE}$	Generalized maximum likelihood estimator of λ
N	Number of crossings
$\hat{\lambda}_{MLE,2}$	Maximum likelihood estimator for λ based on upcrossings
$\hat{\lambda}_{MVUE}$	Minimum variance unbiased estimator for λ

$\hat{\lambda}_b^J$	The Bayes estimator of λ under Jeffery's prior
$\hat{\lambda}_b^W$	The Bayes estimator of λ under Weibull's prior
MSE	Mean squared error
HPDCR	High posterior density credible region

ABSTRACT

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In this thesis we find several classical and Bayesian estimators for the variance of the derivative of a smooth stationary Gaussian process based on its durations and upcrossings above high thresholds. Also, we derive the predictive density of a future duration and number of upcrossings. Moreover, we use a simulation to study the bias and the mean squared errors of the derived estimators. Finally, we apply our findings to meteorological data.

Keywords: Gaussian process; Duration of excursion set; Upcrossings; Bayesian statistics; Grouped data

ملخص

الحامدي، علي احمد عبدالله. الاستدلال الاحصائي لبعض خواص النزوع للعملية الجاوسيه، رسالة ماجستير في العلوم، قسم الاحصاء، جامعة اليرموك2011. (المشرف: الدكتورمحمد طالب العودات).

في هذه الاطروحة نحن أوجدنا عدد من المقدرات الكلاسيكية والبيزيه لتباين مشتقة عملية جاوسية ناعمة ومستقره بالاعتماد على مدات وعدد مرات عبورها فوق مستوى عالي. كذلك قمنا باشتقاق دوال الكثافة الاحتمالية لمدة وعدد مرات العبور المستقبلية. استخدمنا دراسة محاكاه لهذه التقديرات لدراسة الانحياز ومعدل مربع الخطاء لهذه التقديرات. واخيرا قمنا بتطبيق ما اوجدناه علي بيانات حقيقية تمثل سرعات الرياح في احدى محطات الرصد في جمهورية ايرلندا.

الكلمات المفتاحية: العملية الجاوسية، مدات مجموعة النزوع للعملية الجاوسية، عبورالمستوى، النموذج البيزي، بيانات مجمعة

Chapter One

Introduction

Stochastic processes in particular Gaussian processes, are widely used in several fields of science. They have various applications in engineering problems, especially in fields of reliability and safety analysis of structures. It is always necessary and important to evaluate extreme values of the distribution of a random process, since these extreme values represent events of danger or unsafe state of a system. For example, Gaussian processes are used to model several random responses arise in engineering such as the load of a communication system, sea surface elevation and wind speed (Alodat, 2009; Alodat and Anagreh, 2011), where studying extreme values of such random responses is of central interest.

Extreme values assumed by a random process above a given large threshold say u, are considered as a measure assess of reliability or unavailability of the system modeled by that random process. Also the number of times or upcrossings that the process say $X(t)$, crosses the level u is another measure of system reliability. The length of the time interval between an upcrossing and the subsequence downcrossing is also a measure of system unavailability, since it represents the time that the load of a system, modeled by $X(t)$, will spend above u after an upcrossing, see. In Figure 1.1 we give an illustration of upcrossings, downcrossings and durations of a process $X(t)$, above u.

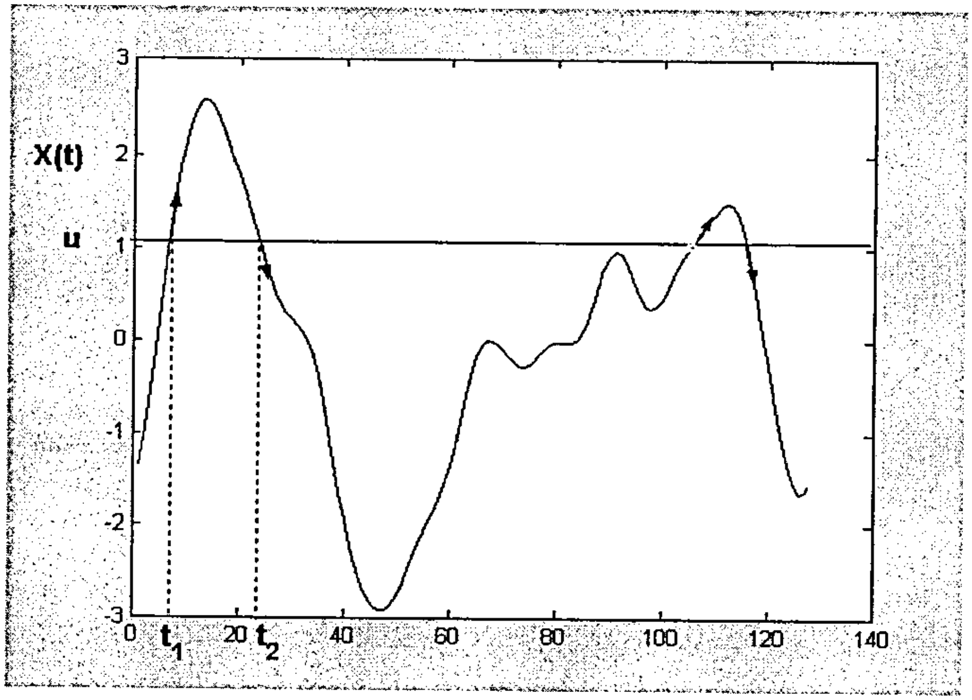

Figure 1.1. Upcrossings ↗, downcrossings ↘, and a duration $s = t_2 - t_1$ of process $X(t)$ above a threshold u.

After extracting these durations from a realization of the sample path of $X(t)$, then we may employ these durations to draw inferences about the processes $X(t)$. However, it is difficult to give a smooth realization of $X(t)$ in practice, i.e., it is not easy to realize $X(t)$ on a smooth grid of [0,T] in some studies. For this reason, extracting the durations of $X(t)$, in [0,T] can not be done accurately. Instead, we are able to observe the durations of a process as grouped data. Hence, if $S_1, \dots, S_N$ represent the durations of $X(t)$, above u in [0,T], then the grouped data of these duration is $\{(I_1, n_1), \dots, (I_{k+1}, n_{k+1})\}$, where $I_1, \dots, I_{k+1}$ is a partition of $(0, \infty)$ and n_j number of $S_j's$ that fall in I_j, $j = 1, \dots, k+1$. This grouped data can be employed to make inference about the process $X(t)$. For example, a climate observatory in Jordan records the wind speeds every four hours through the day, i.e., we have six measurements for the wind speed for each day. If an upcrossing occurs between two consecutive measurements, then it is not possible to know exactly what is the duration of the wind speed process started at that crossings which means that the durations of the wind speed process are observed in practice as grouped data. Making Bayesian statistical inference about a smooth Gaussian process based on its level crossing statistics has not been addressed in literatures as a grouped data problem. Tackling this problem, from grouped data viewpoint, pushes us to develop inferential tools to analyze such data. The rest of this thesis is organized as follows. In chapter two, we introduce the reader to Gaussian processes and their statistics. Also, we state the problem of this research. In chapter three, we derive classical and Bayesian estimators for the variance of the derivative of a smooth Gaussian process. Also, we derive the predictive density of a future duration. Then, we present a simulation study to compare these estimators. In chapter four, classical and Bayesian estimators are derived again, but based on the number of upcrossings of high threshold. Similar to chapter three, predictive densities of future number of upcrossings are also derived. We close the chapter by presenting a simulation study to compare the estimators obtained in this chapter. In chapter five, we apply our findings to real data from the field of meteorology. Finally, we state our conclusion and future work in chapter six.

Chapter Two

Statistics of Gaussian Processes and Literature Review

2.1 Introduction

By a stochastic process, we mean a family $\{X(t), t \in \mathcal{T}\}$ of random variables indexed by a non-empty set $\mathcal{T} \subseteq \mathbb{R}$. For a random process $X(t)$, the set of distributions of the random vectors $(X(t_1), \dots, X(t_n))'$, $t_1, t_2, \dots, t_n \in \mathcal{T}$, n =1, 2, 3,…, are called the finite dimensional distributions of $X(t)$ and they characterize the distribution of $X(t)$.(Schervish, 1995). A stochastic process $X(t)$ is said to be a Gaussian process if for every $t_1, t_2, \dots, t_n$, $n = 1, 2, 3 \dots$, the vector $(X(t_1), \dots, X(t_n))'$ has a multivariate normal distribution. Although every stochastic process is characterized via its finite dimensional distributions, a Gaussian process $X(t)$ is characterized via its mean and covariance functions which are respectively, defined by

$$\mu_X(t) = E(X(t)), \qquad t \in \mathcal{T}.$$

and

$$R_X(t_1, t_2) = Cov(X(t_1), X(t_2)), \qquad t_1, t_2 \in \mathcal{T}.$$

A random process $X(t), t \in \mathcal{T}$ is said to be strict stationary if $(X(t_1), \dots, X(t_n))'$ and $(X(t_1 + \tau), \dots, X(t_n + \tau))'$ have the same distribution for all $n = 1, 2, 3 \dots$, $t_1, \dots, t_n, \tau \in \mathcal{T}$. (Katatbeh et al, 2007).

It is easy to see that a Gaussian process $X(t)$ is stationary iff $\mu_X(t) = \mu_0$ and $R_X(t_1, t_2) = R_X(t_1 - t_2)$, for all $t_1, t_2, t \in \mathcal{T}$. If $X(t)$ is stationary, then it is covariance function is written as $R_X(t)$, where $t = t_1 - t_2$.

In this thesis, we are interested in smooth Gaussian process.

A Gaussian process $X(t), t \in \mathcal{T}$ is said to be differentiable in mean square sense (m.s.) at t, with derivative denoted by $\dot{X}(t)$ if

$$E\left(\frac{X(t+h)-X(t)}{h} - \dot{X}(t)\right)^2 \to 0, \text{ as } h \to 0.$$

(Hwei Hsu, 1997) Also, $X(t)$ is said to be almost surely (a.s.) differentiable at t, if

$$\frac{X(t+h) - X(t)}{h}$$

converges a.s. to some random variable. If $X(t)$ is differentiable in m.s. and a.s. senses, then the two derivatives are equal a.s. So we denote them by $\dot{X}(t)$.

It can be shown that a stationary process $X(t), t \in \mathcal{T}$ is differentiable (in mean square or almost surely) if its covariance function $R_X(t)$ has the following representation (Leadbetter and Spaniolo, 2002)

$$R(t) = \sigma^2 - \frac{\lambda h^2}{2} + o(h^2), \text{ as } h \to 0, \quad (1.1)$$

where $\lambda = Var(\dot{X}(t))$ and $\sigma^2 = Var(X(t))$.

Also, a stationary process $X(t)$ is said to be an ergodic process if for every integrable function $f(x)$, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt = Ef(X(0)) \quad \text{a.s,}$$

where T is the length of $\mathcal{T}$. The ergodicity of $X(t)$ will help us to make statistical inference about the process $X(t)$ based only on the time history of one long realization of $X(t)$. It is well-known result that, a stationary Gaussian process $X(t)$, with variance function $R_X(t)$, is ergodic if $R(t) \to 0$ as $t \to \infty$ (Adler, 1981).

Gaussian processes are widely used in application because their finite dimensional distributions have many good properties such as closure under both marginalization and conditioning. Also, Gaussian processes are closed under differentiation and integration.

In the next section, we present some important statistics of a differentiable Gaussian process. These statistics will be of central interest in this research.

Let $X(t)$, $t \in [0, T]$, be a differentiable Gaussian process. Adler and Taylor (2007) define the excursion set of $X(t)$ above the threshold u as the set of all points $t \in [0, T]$ for which $X(t) \geq u$. If we denote this set by $A(X, u, T)$, then

$$A(X, u, T) = \{\, t \in [0, T]: X(t) \geq u\}.$$

The excursion set of a Gaussian process has been studied extensively in the literature (Adler, 1981; Alodat and Al-Rawwash, 2009; Adler and Taylor, 2007). For large u, we may think of $A(X, u, T)$ as the set of points where the process $X(t)$ assumes extreme values. As $u \to \infty$, Adler (1981) shows that the excursion set of a

Gaussian process decomposes into a finite union of disjoint intervals called clusters or clumps. The lengths of these clusters or clumps are called durations and they are asymptotically independent and identically distributed where the common asymptotic distribution is the same as that of the random variable $S = 2\sqrt{\frac{2Y}{u\lambda}}$, where $Y \sim \exp\left(\frac{1}{u}\right)$. Moreover, the number of these clusters asymptotically follows a Poisson point process with rate α where α is the mean number of local maxima of the process in $[0, T]$ (Adler and Taylor, 2007).

Another important statistic of a random process is the number of upcrossings. The process $X(t)$ is said to have an upcrossing of u at $t_0 \in [0, T]$, if $X(t_0) = u$ and $\dot{X}(t_0) > 0$. If $N(X, u, T)$ denotes the number of such points for a differentiable Gaussian process $X(t)$, then

$$EN(X, u, T) = \frac{T\lambda^{\frac{1}{2}}}{2\pi\sigma}\, e^{-\frac{u^2}{2\sigma^2}}$$

(Leadbetter and Spaniolo, 2002). The random variable $\sup_{t\in[0,T]} X(t)$ has central interest in several applications of processes to engineering. The distribution of this random variable has no closed form in general except for some speed cases. Hence an approximation of $p\{\sup_{t\in[0,T]} X(t) \geq u\}$ could be useful. It was shown that the following approximation is accurate for large u and T.

$$p\{\sup_{t\in[0,T]} X(t) \geq u\} \approx EN(X, u, T).$$

Since $EN(X,u,T)$ is a function of $\lambda^{\frac{1}{2}}$, then estimators of the parameter $\lambda^{\frac{1}{2}}$ are needed. The statistics of a random process can be employed to draw conclusions about the parameter λ. Also, they can be used to construct prediction intervals for future durations or functions of these durations.

The first estimator to $\lambda^{\frac{1}{2}}$ when σ is known was introduce by Rice (1945) which was related to level crossing i.e.,

$$\widehat{\lambda^{\frac{1}{2}}} = \frac{2\pi\sigma}{T} \exp\left(\frac{u^2}{2\sigma^2}\right) N(X,u,T) \text{ .... } (1.2).$$

It is easy to check that $E\left(\widehat{\lambda^{\frac{1}{2}}}\right) = \lambda^{\frac{1}{2}}$.

Holm (1983) used the maximum entropy method to find an estimator for the spectral moment. His estimator has no close form. Cabaña (1985 a, b) proposed an estimator for the second spectral moment of a smooth Gaussian process with known variance based on the values of relative minima and maxima.

Hasofer and Sharpe (1969), Lindgren (1974) and Björnhan and Lindgren (1976) show that

$$\frac{N(X,u,T) - EN(X,u,T)}{\sigma(N(X,u,T))} \xrightarrow{D} N(0,1), T \to \infty,$$

where $\sigma(N(X,u,T))$ is the standard deviation of $N(X,u,T)$. In general $\sigma(N(X,u,T))$ is not available for wide range of processes.

Björnhan and Lindgren (1976) give the following estimators of $\gamma = \left(\frac{\lambda}{\sigma}\right)^{\frac{1}{2}}$.

$$\gamma_U^* = 2\pi \frac{N_u(N(X,u,T))}{T},$$

$$\gamma^{*^2} = \frac{T^{-1}\int_0^\infty \dot{X}(t)^2\,dt}{T^{-1}\int_0^\infty X(t)^2\,dt}, \ldots\ldots\ldots\ldots (1.3)$$

and

$$\gamma_0^* = 2\pi \frac{N_0(N(X,0,T))}{T}.$$

Statistical properties of γ_0^* and γ^* are studied in Lindgren (1974) and Björnhan and Lindgren (1976). They assumed that the mean level is unknown and to be estimated from data. They estimated u by

$$\hat{u} = T^{-1}\int_0^T X(t)dt \ldots\ldots\ldots\ldots (1.4)$$

If the process $X(t)$ is ergodic, then $\hat{u}$ is consistent. Lindgren (1974) has given the estimator $\hat{\lambda}$ for λ:

$$\hat{\lambda} = T^{-1}\int_0^T \dot{X}(t)^2 dt \ldots\ldots\ldots\ldots (1.5)$$

which is approximately an unbiased estimator of λ, i.e., $E(\hat{\lambda}) \approx \lambda$.

Since the estimators (1.3), (1.4) and (1.5) are defined as function of $X(t)$ and $\dot{X}(t)$ then very smooth realizations of $X(t)$ and $\dot{X}(t)$ are needed to find these estimators. Such realization are not available in general.

Lindgren (1974) has combined several estimators of $\lambda^{\frac{1}{2}}$ based on crossings by different levels. His estimator is given by

$$\hat{\lambda} = \hat{\sigma}^2 \frac{2\pi}{3T}\left(N_0\big(N(X,0,T)\big) + \exp\left(\frac{u^2}{2}\right) N_u\big(N(X,u,T)\big) \right.$$

$$\left. + \exp\left(\frac{u^2}{2}\right) N_{-u}\big(N(X,-u,T)\big) \right),$$

where u is chosen such that $u = \frac{2}{3}\hat{\sigma}$, where $\hat{\sigma}$ is an estimator of σ.

Soukissian and Samalekos (2006) have analyzed the durations of the sea level elevations by fitting the extracted durations using a given statistical model. Under the assumption that the sea follows a stationary Gaussian process, they showed that the Weibull distribution fits the durations. In the next chapter, we rely on Bayesian approach to draw inference about the process $X(t)$. Hence, we need to introduce the reader to some tools in Bayesian statistics. In the next section, we present such tools.

2.2 Bayesian Inference

The Bayesian or modern statistics is a branch of statistics which employs prior information together with experimental information to draw statistical inference about a population of interest. This kind of statistics has been used extensively in literatures through the last two decades due to the discovery of markov chain monte- carlo method which removed several computational problems. In this kind of statistics, information from two sources namely experiment and prior knowledge are combined and then to be used in making inference about the parameter of interest. The prior knowledge about a parameter of interest say λ is given by a probability distribution of λ denoted by $\pi(\lambda)$. The distribution $\pi(\lambda)$ describes the degree of belief or our experience about the values of λ before getting the data (Berger, 1985). In the following, we assume that λ is a random variable which has a distribution $\pi(\lambda)$ called a prior distribution of λ. Let Y be a random observation from $f(y|\lambda)$, the conditional distribution of Y given λ.Then, the joint distribution of Y and λ is

$$f(\lambda, y) = f(y|\lambda)\, \pi(\lambda)$$

and the marginal distribution of Y is

$$m(y) = \int f(y|\lambda)\, \pi(\lambda)\, d\lambda.$$

By combining the sample information contained in $f(y|\lambda)$ and the prior information contained in $\pi(\lambda)$, our knowledge about λ can be updated, using the

Bayes rule, via

$$\pi(\lambda|y) = \frac{f(y|\lambda)\,\pi(\lambda)}{m(y)}.$$

The distribution $\pi(\lambda|y)$ is called the posterior distribution of λ. In Bayesian statistics, it is accepted, to various researchers to replace $\pi(\lambda)$ by a non-negative function which makes the integration of $f(y|\lambda)\pi(\lambda)$ finite. Such $\pi(\lambda)$ is called an improper prior for λ. Bayesian inference under improper priors may be interpreted as a weighted likelihood inference. In Bayesian statistics, the loss function is used, instead of mean square error frequents statistics, to quantify on error in a decision about λ. A well-known loss function is the square error loss function which is given by

$$L(\lambda, \vartheta(y)) = (\vartheta(y) - \lambda)^2,$$

where $\vartheta(y)$ is our decision about λ. Also the risk function and the Bayes risk can be used to evaluate decisions about λ. The risk function is

$$R(\lambda, \vartheta(y)) = E_{Y|\lambda} L(\lambda, \vartheta(Y)).$$

and the Bayes risk is

$$r(\pi, \vartheta(y)) = E_\lambda R(\lambda, \vartheta(y)).$$

In Bayesian, the smaller the Bayes risk, the better the estimator. In this thesis, the estimator $\vartheta^*(y)$ which minimizes the Bayes risk or the posterior expected loss

$EL(\lambda, \vartheta(Y))$ is called the Bayes estimator of λ. These concepts of Bayesian statistics will be used in next section by assuming a prior distribution on λ, the smoothness parameter of a Gaussian process $X(t)$, $t \in [0, T]$. So we update our knowledge about λ using the information in $S_1, \ldots, S_N$, where $S_1, \ldots, S_N$ are the observed durations of the process $X(t)$ above a high threshold u. As a prior distribution for λ we will assume that λ has the Gamma distribution with hyper parameters a and b.

Now, assume that we are interested in predicting a future value from $f(y|\lambda)$, say Y. The predictive pdf of Y given the data Y is

$$g(y|x) = \int g(y|\lambda, x)\, \pi(\lambda|y)\, d\lambda.$$

The pdf $g(y|x)$ can be used to make statistical inference about the future value y (Berger, 1985; Bolstad, 2007).

2.3 Statement of the Problem

In this thesis, we assume that we have a differentiable, stationary and ergodic Gaussian process $X(t)$, $t \in [0, T_1]$, with covariance function admitting the representation (1.1). Let N be the number of clusters of the excursion set of $X(t)$ in $[0, T_1]$ a bove large u. Given $N \geq 1$, let $S_1, \ldots, S_N$ denote the durations of these clusters. According to the previous sections, we have the following

$$N = N(X, u, T_1) \sim \text{Poisson}(\alpha),$$

and given $N \geq 1, S_1, \ldots, S_N$ are independent and identically distributed such that

$$S_i \sim f_{S_i}(s|\lambda) = \frac{u^2 \lambda s}{4} \exp\left(-\frac{1}{8} u^2 \lambda s^2\right), s > 0,$$

where α is given in Section 2.

The first step, in this thesis is to find several statistical inferences (classical and Bayesian) about the parameter λ, such as point and interval estimation.

Let M be the number of clusters of the excursion set of $X(t)$ above u in $[T_1, T_2]$ and $V_1, \ldots, V_M$ represent the duration of these clusters. Our second step is to use the data N, $S_1, \ldots, S_N$ to find prediction intervals for one duration V_1 . Since the durations $S_1, \ldots, S_N$ are in general observed only up to intervals, due to lack of a smooth realizations of $X(t)$, then we will rely on the grouped data approach to find inference about λ. The motivation of this proposed research comes from power engineering where the wind speed is used to generate the energy via turbines. At a very high wind speed, turbines should cease power generation in order to be protected from damage. If $X(t)$ denotes the wind speed at time t, the statistics V_{min}, V_{max} represent the minimum and maximum period of power unavailability, the quantity $P = V_1 + \ldots + V_M$ represents the total time of power unavailability in the future time interval $[T_1, T_2]$ and $P_i = \frac{V_i}{P}$ represents the time portion of power unavailability for each duration. Having the distribution of these statistics, we may report some inferential statistics about the reliability of the power generation system.

Chapter Three

Inference Based On Durations

3.1 Introduction

In this chapter, we derive Bayesian estimations for the parameters λ and σ^2 based on a sample of durations $S_1, \dots, S_N$, say. We assume that these durations are observed up to disjoint intervals. Moreover, we derive the predictive density and prediction intervals for future durations. Finally, we conduct a simulation study to look at their performance in term of bias and mean square errors with respect classical counter parts.

3.2 Likelihood Function

For large u, the duration of $X(t)$ above u has the following asymptotic pdf

$$f_{S_1}(s_1) = \frac{u^2 \lambda s_1}{4} \exp\left(-\frac{1}{8} u^2 \lambda {s_1}^2\right), \; s_1 > 0$$

To simplify our analysis, we consider the following transformation

Let $S = \sqrt{W}$. Then $ds = \frac{dw}{2\sqrt{w}}$. The pdf of W is

$$f_W(w) = \frac{u^2 \lambda}{8} \exp\left(-\frac{1}{8} u^2 \lambda w\right), \; w > 0.$$

It is straight forward to show that S and W contain the same amount Fisher information about λ. Hence any inference about λ based on a sample from $f_W(w)$

is equivalent to that based on $f_{S_1}(s_1)$, i.e., the inference will be invariant under the transformation $S = \sqrt{W}$.

Let $I_1 = (0, \delta), \dots, I_j = ((j-1)\delta, j\delta), j = 2, \dots, k$ and $I_{k+1} = (k\delta, \infty)$ be a partition of $(0, \infty)$. Let Y_j denote the number of $W_j's$ that fall in I_j. Conditioning on N=n, we have that $(Y_1, \dots, Y_k) \sim MN(n, p_1, \dots, p_k)$, i.e.,

$$(Y_1, \dots, Y_k | N = n) \sim MN(n, p_1, \dots, p_k),$$

where

$$p_j = \int_{I_j} f_W(w) dw,$$

$$= \exp\left(-\frac{1}{8}u^2\lambda\delta(j-1)\right) - \exp\left(-\frac{1}{8}u^2\lambda\delta j\right), j = 1, 2, \dots, k$$

and

$$p_{k+1} = \exp\left(-\frac{1}{8}u^2\lambda\delta k\right).$$

We calculate the pdf of $Y = (Y_1, \dots, Y_{k+1})$ as follows. Let $\alpha = \frac{T\lambda^{\frac{1}{2}}}{2\pi\sigma} e^{-\frac{u^2}{2\sigma^2}} = c_1 \lambda^{\frac{1}{2}}$, where $c_1 = \frac{T}{2\pi\sigma} e^{-\frac{u^2}{2\sigma^2}}$. Then

$$f(y | N \geq 1, \lambda, \sigma^2) = \frac{\sum_{n=1}^{\infty} P(Y_1 = y_1, \dots, Y_{k+1} = y_{k+1}, N = n)}{P(N \geq 1)},$$

$$= \frac{P(Y_1 = y_1, \dots, Y_{k+1} = y_{k+1}, N = y_1 + \dots + y_{k+1})}{1 - P(N = 0)},$$

$$= \frac{P(Y_1 = y_1, \dots, Y_{k+1} = y_{k+1} | N = y_1 + \dots + y_{k+1})}{1 - \exp(-\alpha)} \times$$

$$P(N = y_1 + \dots + y_{k+1}),$$

Having in mind that Y and N are independent, the last formula simplifies to

$$f(y|N \geq 1, \lambda, \sigma^2) = \frac{(y_1 + \dots + y_{k+1})! \, p_1{}^{y_1} \dots p_{k+1}{}^{y_{k+1}} \exp(-\alpha) \alpha^{y_1 + \dots + y_{k+1}}}{y_1! \dots y_{k+1}! \, (1 - \exp(-\alpha))(y_1 + \dots + y_{k+1})!},$$

$$= \frac{p_1{}^{y_1} \dots p_{k+1}{}^{y_{k+1}} \exp(-\alpha) \alpha^{y_1 + \dots + y_{k+1}}}{y_1! \dots y_{k+1}! \, (1 - \exp(-\alpha))},$$

$$= \frac{p_1{}^{y_1} \dots p_{k+1}{}^{y_{k+1}} \exp(-\alpha) \alpha^{y_1 + \dots + y_{k+1}}}{y_1! \dots y_{k+1}! \, (1 - \exp(-\alpha))}.$$

Finally, formula of $f(y|N \geq 1, \lambda, \sigma^2)$ is given by

$$f(y|N \geq 1, \lambda, \sigma^2) = \frac{\exp(-\alpha)(\alpha p_1)^{y_1} \dots (\alpha p_{k+1})^{y_{k+1}}}{y_1! \dots y_{k+1}! \, (1 - \exp(-\alpha))},$$

$$y_i \in \{0, 1, 2, \dots\}, i = 1, \dots, k+1 \text{ and } y_1 + \dots + y_{k+1} \geq 1.$$

Note that $f(y|N \geq 1, \lambda, \sigma^2)$ can be written as

$$= \frac{\exp(-\alpha p_1)(\alpha p_1)^{y_1} \dots \exp(-\alpha p_{k+1})(\alpha p_{k+1})^{y_{k+1}}}{y_1! \dots y_{k+1}! \, (1 - \exp(-\alpha))},$$

$$= \frac{\exp(-\alpha)\alpha^{y_1+\cdots+y_{k+1}}}{(1-\exp(-\alpha))y_1!\ldots y_{k+1}!} {p_1}^{y_1}\ldots {p_{k+1}}^{y_{k+1}},$$

$$y_i \in \{0,1,2,\ldots\}, i = 1,\ldots,k+1 \text{ and } y_1+\cdots+y_{k+1} \geq 1.$$

By substituting the values of α and $p_1, \ldots, p_{k+1}$ in the last equation and employing the Binomial theorem, $f(y|N \geq 1, \lambda, \sigma^2)$ appears in the following form

$$f(y|N \geq 1,\lambda,\sigma^2) = \frac{{c_1}^{y_1+\cdots+y_{k+1}} \lambda^{\frac{1}{2}\sum_{j=1}^{k+1} y_j} e^{-c_1\lambda^{\frac{1}{2}}}}{\left(1-e^{-c_1\lambda^{\frac{1}{2}}}\right) y_1!\ldots y_{k+1}!} \times$$

$$\exp\left(-\frac{1}{8}u^2\lambda\delta\sum_{j=1}^{k+1}(j-1)\,y_j\right)\left(1-\exp\left(-\frac{1}{8}u^2\lambda\delta\right)\right)^{y_1+\cdots+y_{k+1}},$$

$$= \frac{{c_1}^{y_1+\cdots+y_{k+1}} e^{-c_1\lambda^{\frac{1}{2}}} \lambda^{\frac{1}{2}\sum_{j=1}^{k+1} y_j}}{\left(1-e^{-c_1\lambda^{\frac{1}{2}}}\right) y_1!\ldots y_{k+1}!} \sum_{i=0}^{y_1+\cdots+y_{k+1}} \binom{y_1+\cdots+y_{k+1}}{i} (-1)^i \times$$

$$\exp\left(-\frac{1}{8}u^2\lambda\delta\left(\sum_{j=1}^{k+1}(j-1)\,y_j + i\right)\right),$$

$$y_i \in \{0,1,2,\ldots\}, i = 1,\ldots,k+1 \text{ and } y_1+\cdots+y_{k+1} \geq 1.$$

Let $Z_+ = \{0,1,2,\ldots\}$ and Z_+^{k+1} is the Cartesian product of k copies of Z_+. Also let $\mathcal{L}(Z_+^{k+1})$ be the hyper plane of Z_+^k defined as

$$\mathcal{L}(Z_+^{k+1}) = \{(y_1,\ldots,y_{k+1}) \in Z_+^{k+1} : y_1+\cdots+y_{k+1} \geq 1\}.$$

Now,

the $\mathcal{L}(Z_+^{k+1})$ represents the support of the pdf $f(y|N \geq 1, \lambda, \sigma^2)$. To simplify writing, we introduce the following notation. Let $d = \frac{1}{2}\sum_{j=1}^{k+1} y_j$,

$$A_i(u,k,\delta,y) = \frac{1}{8}u^2\delta\left(\sum_{j=1}^{k+1}(j-1)\,y_j + i\right),$$

and

$$V_i(y) = (-1)^i\binom{2d}{i}.$$

Hence

$$f(y|N \geq 1, \lambda, \sigma^2) = \frac{c_1^{2d}\lambda^d e^{-c_1\lambda^{\frac{1}{2}}}}{\left(1 - e^{-c_1\lambda^{\frac{1}{2}}}\right)}\sum_{i=0}^{2d}\frac{V_i(y)\exp(-A_i(u,k,\delta,y)\lambda)}{y_1!\dots y_{k+1}!},$$

$y \in \mathcal{L}(Z_+^{k+1})$.

In the next section, classical and Bayesian inferences are obtained based on the likelihood $\mathrm{L}(\lambda, \sigma^2) = f(y|N \geq 1, \lambda, \sigma^2)$.

3.3 The Maximum Likelihood Estimator for λ

3.3.1 Case 1. σ^2 is known

Assuming that the parameter σ^2 is known, the MLE for the parameter λ is derived as follows:

$$f(y|N \geq 1, \lambda, \sigma^2) = \frac{\lambda^d e^{-c_1\lambda^{\frac{1}{2}}}}{\left(1 - e^{-c_1\lambda^{\frac{1}{2}}}\right)} \sum_{i=0}^{2d} V_i(y) \exp(-A_i(u,k,\delta,y)\lambda), y \in \mathcal{L}(Z_+^{k+1}).$$

The log likelihood is

$$\log f(y|N \geq 1, \lambda, \sigma^2) = -c_1\lambda^{\frac{1}{2}} - \log\left(1 - e^{-c_1\lambda^{\frac{1}{2}}}\right) + d\log\lambda$$

$$\log\left(\sum_{i=0}^{2d} V_i(y) \exp(-A_i(u,k,\delta,y)\lambda)\right).$$

Taking the derivative of $\log f(y|N \geq 1, \lambda, \sigma^2)$ with respect to λ and setting the derivatives to zero, we find that

$$-\frac{1}{2}c_1\lambda^{-\frac{1}{2}} + \frac{c_1\lambda^{-\frac{1}{2}} e^{-c_1\lambda^{\frac{1}{2}}}}{2\left(1 - e^{-c_1\lambda^{\frac{1}{2}}}\right)} +$$

$$\frac{\sum_{i=0}^{2d} V_i(y) \exp(-A_i(u,k,\delta,y)\lambda)(-A_i(u,k,\delta,y))}{\sum_{i=0}^{2d} V_i(y) \exp(-A_i(u,k,\delta,y)\lambda)} +$$

$$\frac{d\lambda^{-1} \sum_{i=0}^{2d} V_i(y) \exp(-A_i(u,k,\delta,y)\lambda)}{\sum_{i=0}^{2d} V_i(y) \exp(-A_i(u,k,\delta,y)\lambda)} = 0.$$

The numerical solution $\hat{\lambda}$ of the last equation, which satisfies the condition $\partial^2 \log f(y|N \geq 1, \lambda, \sigma^2)|_{\lambda=\hat{\lambda}} < 0$, yields the MLE of λ.

3.3.2 Case 2. λ and σ^2 are unknown

Assuming that the parameters λ and σ^2 are unknown, we the MLE's for the parameters λ and σ^2. The likelihood function of λ, σ^2 is

$L(\lambda, \sigma^2) = f(y|N \geq 1, \lambda, \sigma^2)$, where

$$f(y|N \geq 1, \lambda, \sigma^2) = \frac{\exp\left(-c_2\, \lambda^{\frac{1}{2}} (\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right)}{\left(1 - \exp\left(-c_2\, \lambda^{\frac{1}{2}} (\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right)\right)} \times$$

$$\sum_{i=0}^{2d} V_i(y)\, \lambda^d (\sigma^2)^{-d} e^{-\frac{du^2}{\sigma^2}} \exp(-A_i(u,k,\delta,y)\lambda).$$

Finding the first partial derivatives of $\log f(y|N \geq 1, \lambda, \sigma^2)$ with respect to both λ and σ^2 respectively, and setting these derivatives equal to zero, we find that

$$\log f(y|N \geq 1, \lambda, \sigma^2) = -c_2\, \lambda^{\frac{1}{2}} (\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}} -$$

$$\log\left(1 - \exp\left(-c_2\, \lambda^{\frac{1}{2}} (\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right)\right) +$$

$$\log\left(\sum_{i=0}^{2d} V_i(y) \lambda^d (\sigma^2)^{-d} e^{-\frac{du^2}{\sigma^2}} \exp\left(-A_i(u,k,\delta,y)\lambda\right)\right),$$

where $c_2 = \frac{T}{2\pi}$. The first derivative w.r.t λ is

$$\frac{d}{d\lambda}\log f(y|N \geq 1, \lambda, \sigma^2) = \frac{-c_2\, \lambda^{-\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}}{2} +$$

$$\frac{\exp\left(-c_2\, \lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right) c_2\, \lambda^{-\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}}{2\left(1 - \exp\left(-c_2\, \lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right)\right)} +$$

$$\frac{\sum_{i=0}^{2d} V_i(y) \exp\left(-A_i(u,k,\delta,y)\lambda\right)\left(-A_i(u,k,\delta,y)\right)}{\sum_{i=0}^{2d} V_i(y) \exp\left(-A_i(u,k,\delta,y)\lambda\right)} +$$

$$\frac{\lambda^{-1} d \sum_{i=0}^{2d} V_i(y) \exp\left(-A_i(u,k,\delta,y)\lambda\right)}{\sum_{i=0}^{2d} V_i(y) \exp\left(-A_i(u,k,\delta,y)\lambda\right)} = 0.$$

Taking the derivative with respect to σ^2 and setting it to zero

$$\frac{d}{d\sigma^2}\log f(y|N \geq 1, \lambda, \sigma^2) = \frac{c_2\, \lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{3}{2}} e^{-\frac{u^2}{2\sigma^2}} \left(1 - u^2(\sigma^2)^{-1}\right)}{2} -$$

$$\frac{\exp\left(-c_2\, \lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right) c_2\, \lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{3}{2}} e^{-\frac{u^2}{2\sigma^2}} \left(1 - u^2(\sigma^2)^{-1}\right)}{2\left(1 - \exp\left(-c_2\, \lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right)\right)} +$$

$$\frac{d(\sigma^2)^{-2} u^2 \sum_{i=0}^{2d} V_i(y) \exp\left(-A_i(u,k,\delta,y)\lambda\right)}{\sum_{i=0}^{2d} V_i(y) \exp\left(-A_i(u,k,\delta,y)\lambda\right)} -$$

$$\frac{d(\sigma^2)^{-1}\sum_{i=0}^{2d} V_i(y)\exp\left(-A_i(u,k,\delta,y)\lambda\right)}{\sum_{i=0}^{2d} V_i(y)\exp\left(-A_i(u,k,\delta,y)\lambda\right)} = 0.$$

The numerical solutions $(\hat{\lambda},\widehat{\sigma^2})$ of the last two equations which satisfies the condition $\det\begin{pmatrix} \frac{\partial^2 \log f(y|N\geq 1,\lambda,\sigma^2)}{\partial\lambda^2} & \frac{\partial^2 \log f(y|N\geq 1,\lambda,\sigma^2)}{\partial\lambda\partial\sigma^2} \\ \frac{\partial^2 \log f(y|N\geq 1,\lambda,\sigma^2)}{\partial\lambda\partial\sigma^2} & \frac{\partial^2 \log f(y|N\geq 1,\lambda,\sigma^2)}{\partial\sigma^2} \end{pmatrix}\Big|_{(\lambda,\sigma^2)=(\hat{\lambda},\widehat{\sigma^2})} < 0$ is the MLE's of λ and σ^2.

3.4 Bayes Estimation for λ when σ^2 is known

In this section, Bayes estimators for the parameters λ and σ^2 are derived.

3.4.1 Posterior Distribution of λ

To simplify the work, we may consider the following prior distribution for the parameter λ.

$$\pi(\lambda) = \lambda^r e^{c_1\lambda^{\frac{1}{2}}}\left(1 - e^{-c_1\lambda^{\frac{1}{2}}}\right), \qquad \lambda > 0.$$

Under this prior, the posterior distribution for λ is

$$\pi(\lambda|N\geq 1, y, \sigma^2) \propto \lambda^{d+r}\sum_{i=0}^{2d} V_i(y)\exp(-A_i(u,k,\delta,y)\lambda).$$

Hence

$$\pi(\lambda|N\geq 1, y, \sigma^2) = c\lambda^{d+r}\sum_{i=0}^{2d} V_i(y)\exp(-A_i(u,k,\delta,y)\lambda), \qquad \lambda > 0,$$

where c is a normalizing constant given by

$$c = \left(\sum_{i=0}^{2d} V_i(y) \int_0^{\infty} \lambda^{d+r} \exp\left(-A_i(u,k,\delta,y)\lambda\right) d\lambda \right)^{-1},$$

$$= \left(\sum_{i=0}^{2d} \frac{V_i(y)\Gamma(d+r+1)}{A_i(u,k,\delta,y)^{d+r+1}} \right)^{-1} = \frac{c_1(y)}{\Gamma(d+r+1)},$$

where

$$c_1(y) = \left(\sum_{i=0}^{2d} \frac{V_i(y)}{A_i(u,k,\delta,y)^{d+r+1}} \right)^{-1}.$$

Finally,

$$\pi(\lambda | N \geq 1, y, \sigma^2) = c\lambda^{d+r} \sum_{i=0}^{2d} V_i(y) \exp(-A_i(u,k,\delta,y)\lambda),$$

$$= \frac{c_1(y)\lambda^{d+r}}{\Gamma(d+r+1)} \sum_{i=0}^{2d} V_i(y) \exp(-A_i(u,k,\delta,y)\lambda), \quad \lambda > 0.$$

3.4.2 The Bayes Estimator of λ

The posterior distribution contains all information about λ. Under squared error loss function, the parameter λ is estimated by posterior mean. Let $\hat{\lambda}_{B,1}$ denote the Bayes estimator of λ. Then

$$\hat{\lambda}_{B,1} = E(\lambda|N \geq 1, y, \sigma^2) = \frac{c_1(y)}{\Gamma(d+r+1)} \sum_{i=0}^{2d} V_i(y) \times$$

$$\int_0^\infty \lambda^{d+r+1} \exp(-A_i(u,k,\delta,y)\lambda)\, d\lambda,$$

$$= (d+r+1)c_1(y) \sum_{i=0}^{2d} \frac{V_i(y)}{A_i(u,k,\delta,y)^{d+r+2}}.$$

$$E(\lambda^2|N \geq 1, y, \sigma^2) = \frac{c_1(y)}{\Gamma(d+r+1)} \sum_{i=0}^{2d} V_i(y) \times$$

$$\int_0^\infty \lambda^{d+r+2} \exp(-A_i(u,k,\delta,y)\lambda)\, d\lambda.$$

$$= (d+r+2)(d+r+1)c_1(y) \sum_{i=0}^{2d} \frac{V_i(y)}{A_i(u,k,\delta,y)^{d+r+3}}.$$

Thus, the posterior variance of λ is

$$Var_\pi(\lambda|N \geq 1, y, \sigma^2) = E(\lambda^2|N \geq 1, y, \sigma^2) - \hat{\lambda}_B^2.$$

3.4.3 The Generalized Maximum Likelihood Estimator for λ

In this section, we find the generalized maximum likelihood estimator of λ. To find the GMLE of λ, we take, the log likelihood of $\pi(\lambda|N \geq 1, y, \sigma^2)$.

i.e.,

$$\log \pi(\lambda|N \geq 1, y, \sigma^2) = \log\left(\frac{c_1(y)}{\Gamma(d+r+1)} \sum_{i=0}^{2d} V_i(y)\lambda^{d+r} \exp(-A_i(u,k,\delta,y)\lambda)\right),$$

Take the first derivative of $\log \pi(\lambda|N \geq 1, y, \sigma^2)$ with respect to λ, we get

$$\frac{d}{d\lambda} \log \pi(\lambda|N \geq 1, y, \sigma^2) = \frac{\sum_{i=0}^{2d} V_i(y) B_i(u,k,\delta,y,\lambda)}{\lambda^{d+r} \sum_{i=0}^{2d} V_i(y) \exp(-A_i(u,k,\delta,y)\lambda)},$$

where

$$B_i(u,k,\delta,y,\lambda) = -A_i(u,k,\delta,y)\lambda^{d+r} \exp(-A_i(u,k,\delta,y)\lambda) + (d+r)\lambda^{d+r-1} \exp(-A_i(u,k,\delta,y)\lambda).$$

The GMLE of λ, denoted it by $\hat{\lambda}_{GMLE}$, is the solution of the equation of λ.

$$\frac{\sum_{i=0}^{2d} V_i(y) B_i(u,k,\delta,y,\lambda)}{\lambda^{d+r} \sum_{i=0}^{2d} V_i(y) \exp(-A_i(u,k,\delta,y)\lambda)} = 0.$$

3.5 Bayes Estimation for λ and σ^2

In this section, it is assumed that both parameters λ and σ^2 are unknown. Under this assumption, classical and Bayesian estimator are derived. Also, we derive prediction intervals for a future duration. Since λ and σ^2 are unknown, then we need to rewrite the pdf $f(y|N \geq 1, \lambda, \sigma^2)$ in the following form.

$$f(y|N \geq 1, \lambda, \sigma^2) = \frac{\exp\left(-\frac{T\lambda^{\frac{1}{2}}}{2\pi\sigma} e^{-\frac{u^2}{2\sigma^2}}\right)\left(\frac{T\lambda^{\frac{1}{2}}}{2\pi\sigma} e^{-\frac{u^2}{2\sigma^2}}\right)^{2d}}{\left(1 - \exp\left(-\frac{T\lambda^{\frac{1}{2}}}{2\pi\sigma} e^{-\frac{u^2}{2\sigma^2}}\right)\right) y_1! \ldots y_{k+1}!} \times$$

$$\exp\left(-\frac{1}{8}u^2\lambda\delta\sum_{j=1}^{k+1}(j-1)\,y_j\right)\left(1-\exp\left(-\frac{1}{8}u^2\lambda\delta\right)\right)^{2d},$$

$y \in \mathcal{L}(Z_+^{k+1})$.

To simplify, we introduce the following notation. Let $c_2 = \frac{T}{2\pi}$ and

$$V_i(y) = (-1)^i \binom{2d}{i}.$$

Hence

$$f(y|N \geq 1, \lambda, \sigma^2) = \frac{\exp\left(-c_2\,\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}}\right)\left(\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}}\right)^{2d}}{\left(1-\exp\left(-c_2\,\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}}\right)\right)y_1!\ldots y_{k+1}!}\times$$

$$\exp\left(-\frac{1}{8}u^2\lambda\delta\sum_{j=1}^{k+1}(j-1)\,y_j\right)\left(1-\exp\left(-\frac{1}{8}u^2\lambda\delta\right)\right)^{2d},$$

$y \in \mathcal{L}(Z_+^{k+1})$.

Using the Binomial theorem, we can write $f(y|N \geq 1, \lambda, \sigma^2)$ as follows:

$$f(y|N \geq 1, \lambda, \sigma^2) = \frac{c_2^{2d}\exp\left(-c_2\,\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}}\right)}{\left(1-\exp\left(-c_2\,\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}}\right)\right)}\times$$

$$(\sigma^2)^{-d}e^{-\frac{du^2}{\sigma^2}}\lambda^d\sum_{i=0}^{2d}\frac{V_i(y)\exp(-A_i(u,k,\delta,\boldsymbol{y})\lambda)}{y_1!\ldots y_{k+1}!},$$

$y \in \mathcal{L}(Z_+^{k+1})$.

To simplify the work, we may consider the following joint prior distribution for the parameters λ and σ^2.

$$\pi(\lambda, \sigma^2) = \frac{\lambda^{r_1}\left(1 - \exp\left(-c_2\, \lambda^{\frac{1}{2}} (\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right)\right)}{(\sigma^2)^{r_2} \exp\left(-c_2\, \lambda^{\frac{1}{2}} (\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right)}, \qquad \sigma^2 > 0, \lambda > 0.$$

3.5.1 Posterior of λ and σ^2

Under this prior, the joint posterior distribution for λ and σ^2 satisfies

$$\pi(\lambda, \sigma^2 | N \geq 1, y) \propto (\sigma^2)^{-d-r_2} e^{-\frac{du^2}{\sigma^2}} \sum_{i=0}^{2d} V_i(y)\, \lambda^{d+r_1} \exp(-A_i(u, k, \delta, y)\lambda).$$

Hence

$$\pi(\lambda, \sigma^2 | N \geq 1, y) = c(y) \times$$

$$\lambda^{d+r_1} (\sigma^2)^{-d-r_2} e^{-\frac{du^2}{\sigma^2}} \sum_{i=0}^{2d} V_i(y) \exp(-A_i(u, k, \delta, y)\lambda),$$

where

$$c(y) = \left(\sum_{i=0}^{2d} V_i(y) \int_0^\infty \left((\sigma^2)^{-d-r_2} e^{-\frac{du^2}{\sigma^2}} \int_0^\infty \lambda^{d+r_1} e^{-A_i(u,k,\delta,y)\lambda} d\lambda\right) d\sigma^2\right)^{-1},$$

$$= \left(\sum_{i=0}^{2d} V_i(\boldsymbol{y}) \frac{\Gamma(d + r_1 + 1)}{A_i(u, k, \delta, \boldsymbol{y})^{d+r_1+1}} \int_0^{\infty} (\sigma^2)^{-d-r_2} e^{-\frac{du^2}{\sigma^2}} d\sigma^2 \right)^{-1}.$$

$$c(\boldsymbol{y}) = \left(\frac{\Gamma(d + r_1 + 1)\Gamma(d + r_2 - 1)}{(du^2)^{d+r_2-1}} \sum_{i=0}^{2d} \frac{V_i(\boldsymbol{y})}{A_i(u, k, \delta, \boldsymbol{y})^{d+r_1+1}} \right)^{-1},$$

$$= \frac{(du^2)^{d+r_2-1}}{\Gamma(d + r_1 + 1)\Gamma(d + r_2 - 1)} \left(\sum_{i=0}^{2d} \frac{V_i(\boldsymbol{y})}{A_i(u, k, \delta, \boldsymbol{y})^{d+r_1+1}} \right)^{-1},$$

$$= \frac{c_1(\boldsymbol{y})(du^2)^{d+r_2-1}}{\Gamma(d + r_1 + 1)\Gamma(d + r_2 - 1)}.$$

where

$$c_1(\boldsymbol{y}) = \left(\sum_{i=0}^{2d} \frac{V_i(\boldsymbol{y})}{A_i(u, k, \delta, \boldsymbol{y})^{d+r_1+1}} \right)^{-1}.$$

Finally, the joint posterior distribution of λ and σ^2 is

$$\pi(\lambda, \sigma^2 | N \geq 1, \boldsymbol{y}) = \frac{c_1(\boldsymbol{y})(du^2)^{d+r_2-1} \lambda^{d+r_1} (\sigma^2)^{-d-r_2} e^{-\frac{du^2}{\sigma^2}}}{\Gamma(d + r_1 + 1)\Gamma(d + r_2 - 1)} \times$$

$$\sum_{i=0}^{2d} V_i(\boldsymbol{y}) \exp(-A_i(u, k, \delta, \boldsymbol{y})\lambda).$$

3.5.2 The Marginal Posterior Density of λ

The marginal distribution of λ is obtained by integrating the joint pdf with respect to σ^2.

Integrating with respect to σ^2 yeilds

$$\pi(\lambda|N \geq 1, y) = \frac{c_1(y)(du^2)^{d+r_2-1}\lambda^{d+r_1}}{\Gamma(d+r_1+1)\Gamma(d+r_2-1)} \times$$

$$\sum_{i=0}^{2d} V_i(y)\exp(-A_i(u,k,\delta,y)\lambda)\int_0^{\infty}(\sigma^2)^{-d-r_2}e^{-\frac{du^2}{\sigma^2}}d\sigma^2,$$

$$= \frac{c_1(y)\lambda^{d+r_1}}{\Gamma(d+r_1+1)}\sum_{i=0}^{2d} V_i(y)\exp(-A_i(u,k,\delta,y)\lambda),$$

$$= \sum_{i=0}^{2d}\frac{c_1(y)V_i(y)}{A_i(u,k,\delta,y)^{d+r_1+1}}\pi_i(\lambda) = \sum_{i=0}^{2d} w_i\,\pi_i(\lambda),$$

where

$$w_i = \frac{V_i(y)/A_i(u,k,\delta,y)^{d+r_1+1}}{\sum_{j=0}^{2d}\frac{V_j(y)}{A_j(u,k,\delta,y)^{d+r_1+1}}}$$

and

$\pi_i(\lambda)$ is the pdf of Gamma $\left(d+r+1, \frac{1}{A_i(u,k,\delta,y)}\right)$.

The Bayes estimator of λ is

$$\hat{\lambda}_{B,2} = E(\lambda|N \geq 1, \boldsymbol{y}) = \frac{c_1(\boldsymbol{y})}{\Gamma(d + r_1 + 1)} \sum_{i=0}^{2d} V_i(\boldsymbol{y}) \times$$

$$\int_0^{\infty} \lambda^{d+r_1+1} \exp(-A_i(u, k, \delta, \boldsymbol{y})\lambda) \, d\lambda,$$

$$= (d + r_1 + 1) \sum_{i=0}^{2d} \frac{w_i}{A_i(u, k, \delta, \boldsymbol{y})}.$$

$$E\left(\lambda^2|N \geq 1, \boldsymbol{y}\right) = \frac{c_1(\boldsymbol{y})}{\Gamma(d + r_1 + 1)} \sum_{i=0}^{2d} V_i(\boldsymbol{y}) \times$$

$$\int_0^{\infty} \lambda^{d+r_1+2} \exp(-A_i(u, k, \delta, \boldsymbol{y})\lambda) \, d\lambda,$$

$$= (d + r_1 + 1)(d + r_1 + 2) \sum_{i=0}^{2d} \frac{w_i}{A_i(u, k, \delta, \boldsymbol{y})^2}.$$

Hence

$$Var(\lambda|N \geq 1, \boldsymbol{y}) = (d + r_1 + 1)(d + r_1 + 2) \sum_{i=0}^{2d} \frac{w_i}{A_i(u, k, \delta, \boldsymbol{y})^2} -$$

$$(d + r_1 + 1)^2 \left(\sum_{i=0}^{2d} \frac{w_i}{A_i(u, k, \delta, \boldsymbol{y})}\right)^2.$$

3.5.3 The Marginal Posterior Density of σ^2

Following similar argument as in the previous section, the marginal distribution of σ^2 is obtain by integrating joint pdf with respect to λ. Integrating λ out, we get

$$\pi(\sigma^2|N \geq 1, y) = \frac{c_1(y)(du^2)^{d+r_2-1}(\sigma^2)^{-d-r_2}e^{-\frac{du^2}{\sigma^2}}}{\Gamma(d+r_1+1)\Gamma(d+r_2-1)}\sum_{i=0}^{2d} V_i(y) \times$$

$$\int_0^{\infty} \lambda^{d+r_1}\exp\left(-A_i(u,k,\delta,y)\lambda\right) d\lambda,$$

$$= \frac{c_1(y)(du^2)^{d+r_2-1}(\sigma^2)^{-d-r_2}e^{-\frac{du^2}{\sigma^2}}}{\Gamma(d+r_2-1)}\sum_{i=0}^{2d}\frac{V_i(y)}{A_i(u,k,\delta,y)^{d+r_1+1}}.$$

$$= \sum_{i=0}^{2d}\frac{c_1(y)V_i(y)}{A_i(u,k,\delta,y)^{d+r_1+1}}\pi_i(\sigma^2) = \sum_{i=0}^{2d} w_i\,\pi_i(\sigma^2),$$

where

$$w_i = \frac{V_i(y)/A_i(u,k,\delta,y)^{d+r_1+1}}{\sum_{j=0}^{2d}\frac{V_j(y)}{A_j(u,k,\delta,y)^{d+r_1+1}}}$$

and

$\pi_i(\sigma^2)$ is the pdf of IG $\left(d+r_2-1, \frac{1}{du^2}\right)$.

The Bayes estimator of σ^2 is

$$\widehat{\sigma^2}_{B,1} = E(\sigma^2|N \geq 1, y) = \frac{c_1(y)(du^2)^{d+r_2-1}}{\Gamma(d+r_2-1)}\sum_{i=0}^{2d}\frac{V_i(y)}{A_i(u,k,\delta,y)^{d+r_1+1}} \times$$

$$\int_0^\infty (\sigma^2)^{-d-r_2+1} e^{-\frac{du^2}{\sigma^2}}\, d\sigma^2,$$

$$= du^2(d+r_2-1)\sum_{i=0}^{2d} \frac{c_1(y)V_i(y)}{A_i(u,k,\delta,\boldsymbol{y})^{d+r_1+1}}.$$

$$E\left((\sigma^2)^2 | N \geq 1, \boldsymbol{y}\right) = \frac{(du^2)^{d+r_2-1}}{\Gamma(d+r_2-1)} \sum_{i=0}^{2d} \frac{c_1(y)V_i(y)}{A_i(u,k,\delta,\boldsymbol{y})^{d+r_1+1}} \times$$

$$\int_0^\infty (\sigma^2)^{-d-r_2+2} e^{-\frac{du^2}{\sigma^2}}\, d\sigma^2,$$

$$= \frac{(du^2)^2}{(d+r_2-2)(d+r_2-3)} \sum_{i=0}^{2d} \frac{c_1(y)V_i(y)}{A_i(u,k,\delta,\boldsymbol{y})^{d+r_1+1}}.$$

$$Var(\sigma^2 | N \geq 1, \boldsymbol{y}) = \frac{(du^2)^2}{(d+r_2-2)(d+r_2-3)} \sum_{i=0}^{2d} \frac{c_1(y)V_i(y)}{A_i(u,k,\delta,\boldsymbol{y})^{d+r_1+1}} -$$

$$du^4(d+r_2-1)^2 \left(\sum_{i=0}^{2d} \frac{c_1(y)V_i(y)}{A_i(u,k,\delta,\boldsymbol{y})^{d+r_1+1}}\right)^2.$$

3.6 Predictive Density of Future Duration

3.6.1 Case1. σ^2known

Let S_1 be a future duration of $X(t)$ above high level u in [T, T +T_1], T,T_1 > 0. Also, Let M be the number of durations of $X(t)$ above u in [T, T +T_1]. To find the predictive density of S_1 given y and $N \geq 1, M \geq 1$, we first find the predictive density of $S = \sqrt{W}$ given y and $N \geq 1, M \geq 1$. The predictive density of W given y is

$$g(w|N \geq 1, M \geq 1, y) = \int_0^\infty f(w|\lambda)\, \pi(\lambda|N \geq 1, M \geq 1, y, \sigma^2) d\lambda$$

$$= \frac{u^2 c_1(y)}{8\Gamma(d+r+1)} \exp\left(-\frac{1}{8}u^2\lambda w\right) \times$$

$$\sum_{i=0}^{2d} V_i(y) \int_0^\infty \lambda^{d+r+1} \exp(-A_i(u,k,\delta,y)\lambda)\, d\lambda,$$

$$= \frac{c_1(y)u^2}{8\Gamma(d+r+1)} \sum_{i=0}^{2d} V_i(y) \times$$

$$\int_0^\infty \lambda^{d+r+1} \exp\left(-\left(A_i(u,k,\delta,y) + \frac{1}{8}u^2 w\right)\lambda\right) d\lambda.$$

Hence,

$$g(w|N \geq 1, M \geq 1, y) = \frac{c_1(y)u^2(d+r+1)}{8} \sum_{j=0}^{2d} \frac{V_j(y)}{\left(A_j(u,k,\delta,y) + \frac{1}{8}u^2 w\right)^{d+r+2}}.$$

The CDF of $g(w|N \geq 1, M \geq 1, y)$ is given by

$G(w|N \geq 1, M \geq 1, y) = 0$ for $w < 0$ and for $w \geq 0$,

$$G(w|N \geq 1, M \geq 1, y) = c_1(y) \sum_{j=0}^{2d} V_j(y) \times$$

$$\left(\frac{1}{A_j(u,k,\delta,y)^{d+r+1}} - \frac{1}{\left(A_j(u,k,\delta,y) + \frac{1}{8}u^2 w\right)^{d+r+1}} \right).$$

The solution of $g(L|N \geq 1, M \geq 1, y)=\alpha_1$ and $g(U|N \geq 1, M \geq 1, y)=1-\alpha_2$, where $\alpha_1, \alpha_2 > 0$ and $\alpha = \alpha_1 + \alpha_2$ is a 100 (1-α)% prediction interval of W.

A 100 (1-α)% prediction interval for W is $[L, U]$, where L is the solution of

$$\int_0^L g(w|N \geq 1, M \geq 1, y)dw = \alpha_1,$$

And U is the solution of

$$\int_U^\infty g(w|N \geq 1, M \geq 1, y)dw = \alpha_2,$$

Hence the 100 (1-α)% for S_1 is $[L^2, U^2]$. The mean of $g(w|N \geq 1, M \geq 1, y)$ serves as a prediction of W. Hence, $\widehat{W}$, the prediction of W is

$$E(S_1^2|y) = E(W|N \geq 1, M \geq 1, y) = \int_0^\infty w g(w|N \geq 1, M \geq 1, y)dw,$$

$$= \frac{c_1(y)u^2(d+r+1)}{8} \sum_{j=0}^{2d} V_j(y) \int_0^\infty \frac{wdw}{\left(A_j(u,k,\delta,y) + \frac{1}{8}u^2 w\right)^{d+r+2}},$$

$$= \frac{8c_1(y)(d+r+1)}{u^2}\sum_{j=0}^{2d} V_j(y)\int_{A_j(u,k,\delta,y)}^{\infty} \frac{z - A_j(u,k,\delta,y)}{z^{d+r+2}} dz,$$

$$= \frac{8c_1(y)(d+r+1)}{u^2}\sum_{j=0}^{2d} V_j(y) \times$$

$$\left(\frac{1}{(d+r)A_j(u,k,\delta,y)^{d+r}} - \frac{A_j(u,k,\delta,y)}{(d+r+1)A_j(u,k,\delta,y)^{d+r+1}}\right),$$

$$= \frac{8c_1(y)(d+r+1)}{u^2}\sum_{j=0}^{2d} \frac{V_j(y)}{(d+r)(d+r+1)A_j(u,k,\delta,y)^{d+r}},$$

$$= \frac{8c_1(y)}{u^2(d+r)}\sum_{j=0}^{2d} \frac{V_j(y)}{A_j(u,k,\delta,y)^{d+r}}.$$

The $E(W^2|N \geq 1, M \geq 1, y)$ serves as a prediction of S_1. Hence

$$E(S_1|y) = E\left(W^{\frac{1}{2}}\middle|N \geq 1, M \geq 1, y\right) = \int_0^{\infty} w^{\frac{1}{2}} g(w|N \geq 1, M \geq 1, y)dw,$$

$$= \frac{c_1(y)u^2(d+r+1)}{8}\sum_{j=0}^{2d} V_j(y)\int_0^{\infty} \frac{w^{\frac{1}{2}}dw}{\left(A_j(u,k,\delta,y) + \frac{1}{8}u^2 w\right)^{d+r+2}},$$

$$= \frac{c_1(y)u^2(d+r+1)}{8}\sum_{j=0}^{2d} \frac{V_j(y)8\sqrt{2\pi}\Gamma(d+r+\frac{1}{2})}{u^3 A_j(u,k,\delta,y)^{d+r+\frac{1}{2}}\Gamma(d+r+2)},$$

$$= \frac{c_1(y)\sqrt{2\pi}\Gamma(d+r+\frac{1}{2})}{u\Gamma(d+r+1)}\sum_{j=0}^{2d} \frac{V_j(y)}{A_j(u,k,\delta,y)^{d+r+\frac{1}{2}}}.$$

Hence

$$Var(S_1|N\geq 1, M\geq 1, \boldsymbol{y}) = E(S_1^2|N\geq 1, M\geq 1, \boldsymbol{y}) - \left(E(S_1|N\geq 1, M\geq 1, \boldsymbol{y})\right)^2.$$

3.6.2 Case2. σ^2 unknown

$$g(w|N\geq 1, M\geq 1, \boldsymbol{y}) = \int_0^\infty\int_0^\infty f(w|\lambda)\,\pi(\lambda,\sigma^2|N\geq 1, M\geq 1, \boldsymbol{y})d\lambda d\sigma^2$$

$$= \frac{u^2 c_1(\boldsymbol{y})(du^2)^{d+r_2-1}}{8\Gamma(d+r_1+1)\Gamma(d+r_2-1)}\sum_{i=0}^{2d} V_i(\boldsymbol{y})\times$$

$$\int_0^\infty\left((\sigma^2)^{-d-r_2}e^{-\frac{du^2}{\sigma^2}}\int_0^\infty \lambda^{d+r_1+1}\exp\left(-\left(A_i(u,k,\delta,\boldsymbol{y})+\frac{1}{8}u^2w\right)\lambda\right)d\lambda\right)d\sigma^2,$$

$$= \frac{u^2(d+r_1+1)c_1(\boldsymbol{y})(du^2)^{d+r_2-1}}{8\Gamma(d+r_2-1)}\times$$

$$\sum_{j=0}^{2d}\frac{V_i(\boldsymbol{y})}{\left(A_j(u,k,\delta,\boldsymbol{y})+\frac{1}{8}u^2w\right)^{d+r_1+2}}\int_0^\infty(\sigma^2)^{-d-r_2}e^{-\frac{du^2}{\sigma^2}}d\sigma^2,$$

Finally

$$g(w|N\geq 1, M\geq 1, \boldsymbol{y}) = \frac{c_1(\boldsymbol{y})u^2(d+r_1+1)}{8}\sum_{j=0}^{2d}\frac{V_j(\boldsymbol{y})}{\left(A_j(u,k,\delta,\boldsymbol{y})+\frac{1}{8}u^2w\right)^{d+r_1+2}}.$$

$$E(S_1^2|N\geq 1, M\geq 1, \boldsymbol{y}) = E(W|N\geq 1, M\geq 1, \boldsymbol{y}) = \int_0^\infty wg(w|N\geq 1, M\geq 1, \boldsymbol{y})dw$$

$$= \frac{c_1(\boldsymbol{y})u^2(d+r_1+1)}{8}\sum_{j=0}^{2d}V_j(\boldsymbol{y})\int_0^\infty\frac{wdw}{\left(A_j(u,k,\delta,\boldsymbol{y})+\frac{1}{8}u^2w\right)^{d+r_1+2}},$$

$$= 8(d+r_1+1)c_1(\boldsymbol{y})\sum_{j=0}^{2d}V_j(\boldsymbol{y})\times$$

$$\left(\frac{1}{dA_j(u,k,\delta,\boldsymbol{y})^{d+r_1}}-\frac{A_j(u,k,\delta,\boldsymbol{y})}{(d+1)A_j(u,k,\delta,\boldsymbol{y})^{d+r_1+1}}\right),$$

$$=8(d+r_1+1)c_1(\boldsymbol{y})\sum_{j=0}^{2d}\frac{V_j(\boldsymbol{y})}{(d+r_1)(d+r_1+1)A_j(u,k,\delta,\boldsymbol{y})^{d+r_1}},$$

$$=\frac{8c_1(\boldsymbol{y})}{(d+r_1)}\sum_{j=0}^{2d}\frac{V_j(\boldsymbol{y})}{A_j(u,k,\delta,\boldsymbol{y})^{d+r_1}}.$$

$$E(S_1|N\geq 1,M\geq 1,\boldsymbol{y})=E\left(W^{\frac{1}{2}}\middle|N\geq 1,M\geq 1,\boldsymbol{y}\right)$$

$$=\int_0^\infty w^{\frac{1}{2}}g(w|N\geq 1,M\geq 1,\boldsymbol{y})dw$$

$$E(S_1|\boldsymbol{y})=\frac{c_1(\boldsymbol{y})u^2(d+r_1+1)}{8}\sum_{j=0}^{2d}V_j(\boldsymbol{y})\int_0^\infty\frac{w^{\frac{1}{2}}dw}{\left(A_j(u,k,\delta,\boldsymbol{y})+\frac{1}{8}u^2w\right)^{d+r_1+2}},$$

$$=\frac{c_1(\boldsymbol{y})u^2(d+r_1+1)}{8}\sum_{j=0}^{2d}\frac{V_j(\boldsymbol{y})8\sqrt{2\pi}\Gamma(d+r_1+\frac{1}{2})}{u^3A_j(u,k,\delta,\boldsymbol{y})^{d+r_1+\frac{1}{2}}\Gamma(d+r_1+2)},$$

$$=\frac{c_1(\boldsymbol{y})\sqrt{2\pi}\Gamma(d+r_1+\frac{1}{2})}{u\Gamma(d+r_1+1)}\sum_{j=0}^{2d}\frac{V_j(\boldsymbol{y})}{A_j(u,k,\delta,\boldsymbol{y})^{d+r_1+\frac{1}{2}}}.$$

$$Var(S_1|N\geq 1,M\geq 1,\boldsymbol{y})=E(S_1^2|N\geq 1,M\geq 1,\boldsymbol{y})-\left(E(S_1|N\geq 1,M\geq 1,\boldsymbol{y})\right)^2.$$

3.7 Simulation

In the section, we conduct a simulation study to compare the bias and mean squared errors for the estimators obtained in the previous sections for the case when σ is known. For each estimator, we design the following algorithm to find the bias and the mean squared error of an estimator $\hat{\lambda}$:

1. Input values $\lambda, u, T, k, \delta, \sigma$.
2. Find $p_1, \dots, p_{k+1}$ and $\alpha = c_1 \lambda^{\frac{1}{2}}$.
3. Simulate N from Poisson (α).
4. Simulate $Y_1, \dots, Y_{k+1}$ from $MN(n, p_1, \dots, p_{k+1})$.
5. Find the value of the estimator according to its definition.
6. Repeat the steps (2) – (5) L times to get L the values of the estimators $\hat{\lambda}_1, \dots, \hat{\lambda}_L$.
7. Compute bias and mean square error of the estimator as follows:

   bias $= \frac{1}{L}\sum_{i=1}^{L}(\hat{\lambda}_i - \lambda)$ and

   mse $= \frac{1}{L-1}\sum_{i=1}^{L}(\hat{\lambda}_i - \lambda)^2 + \text{bias}^2$

This algorithm has been implemented for the following different values of λ, u, T, k, δ and σ which are given in Table 3.1.

Table 3.1 Values of $\sigma, \lambda, \delta, T,$ and k used in simulation

σ	1 2
λ	1 2
δ	0.5 1 1.5 2
T	250 300 350
k	2 3 4 5

Since the sample size N is random and is observed from the experiment, then it is difficult to do a strict comparison between the two estimators, since the bias and MSE of both estimators $\hat{\lambda}_{B,1}$ and $\hat{\lambda}_{MLE,1}$ are influenced by the sample size.

However, we may compare the bias and MSE as functions of the average sample size. Since the average sample size is a linear function in T, then it is reasonable to compare the biases and MSE's of estimators as functions in T. The results of simulation are presented in Tables (3.2)-(3.7). Based on these tables we may report the following concluding remarks.

1. The bias and MSE are decreasing functions in T, for fixed δ, σ, k and λ.
2. The bias and MSE are in general decreasing as a function in δ, for fixed T, λ, σ, k.
3. The bias and MSE are in general decreasing as function in k, fixed T, λ, σ and δ.
4. When comparing Bayes estimator and MLE, it can to be seen that the MLE is better in general than the Bayes estimator in term of bias and MSE.

Based on the Tables (3.8)-(3.13) we write the following comments

1. For the bias and MSE of the two both estimators we don't find a clear pattern as function in k, fixed δ, σ, T and λ.
2. We note from Tables (3.8)-(3.13) that the Bayes estimator is better than the MLE. This can be interpreted as follows since $\lambda = 2$, is large, then the duration distribution will assume small values. In this case the majority of observed durations will fall in the first interval. Hence, small a mount of information about λ will be contained in the grouped data, while the Bayes estimator uses the prior information to increase the amount of information about λ.
3. Its clear that Bayes estimator is better than MLE when λ, σ are increasing for values k, for fixed T.

Table 3.2 Bias and mean squared errors for $\hat{\lambda}_{B,1}$, T=200, $\sigma = 1, \lambda = 1$

K		2		3		4		5	
		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	1.98095	4.68669	1.17357	1.94374	0.755066	0.681885	0.505289	0.381552
	1	0.834278	1.3178	0.429361	0.50988	0.225627	0.38534	0.110028	0.322632
	1.5	0.484295	0.938972	0.200962	0.334476	0.105816	0.496984	0.021886	0.289539
	2	0.29534	0.602189	0.185771	0.778195	0.070593	0.785895	0.051685	0.74225

Table 3.3 Bias and mean squared errors for MLE, T=250 , $\sigma = 1, \lambda = 1$

K		2		3		4		5	
		MLE		MLE		MLE		MLE	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	1.12487	1.71107	0.752347	0.863919	0.457831	0.463542	0.267081	0.291267
	1	0.639361	0.526975	0.400288	0.257689	0.250525	0.157238	0.157578	0.119693
	1.5	0.380767	0.224014	0.218349	0.118416	0.136542	0.09213	0.082909	0.074006
	2	0.24144	0.129276	0.124154	0.078773	0.081036	0.076385	0.052675	0.072709

Table 3.4 Bias and mean squared errors for $\hat{\lambda}_{B,1}$, T=300, $\sigma = 1, \lambda = 1$

	K	2		3		4		5	
		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	1.84681	3.56109	1.06439	1.22626	0.672204	0.531245	0.432708	0.260088
	1	0.746122	0.667616	0.342015	0.200509	0.152858	0.166779	0.045214	0.079423
	1.5	0.420058	0.272648	0.16278	0.218212	0.034576	0.103625	-0.02148	0.110874
	2	0.279068	0.244665	0.119352	0.365403	0.010657	0.133073	-0.01726	0.73105

Table 3.5 Bias and mean squared errors for MLE, T=300, $\sigma = 1, \lambda = 1$

	K	2		3		4		5	
		MLE		MLE		MLE		MLE	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	0.715761	1.1043	0.411658	0.579243	0.224219	0.379402	0.088331	0.27723
	1	0.559443	0.463582	0.284288	0.217971	0.150874	0.134842	0.053794	0.109981
	1.5	0.369026	0.205064	0.215073	0.11783	0.095421	0.075318	0.03432	0.073175
	2	0.229436	0.113143	0.116041	0.068526	0.078708	0.06967	0.037537	0.059275

Table 3.6 Bias and mean squared errors for $\hat{\lambda}_{B,1}$, T=350, $\sigma = 1, \lambda = 1$

	K	2		3		4		5	
		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	1.7459	3.15877	0.989905	1.05516	0.609834	0.431875	0.379073	0.19572
	1	0.690649	0.558567	0.301618	0.156973	0.109612	0.067863	0.002727	0.050694
	1.5	0.380047	0.237346	0.129395	0.099633	0.006158	0.123947	-0.05985	0.084543
	2	0.256688	0.243171	0.077666	0.093745	-0.00596	0.803301	-0.01007	0.166179

Table 3.7 Bias and mean squared errors for MLE, T=350, $\sigma = 1, \lambda = 1$

	K	2		3		4		5	
		MLE		MLE		MLE		MLE	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	0.247205	0.624917	0.130373	0.417233	0.134641	0.37578	0.028234	0.25511
	1	0.357307	0.362441	0.113793	0.179012	-0.01732	0.129875	-0.04719	0.93184
	1.5	0.317763	0.199753	0.101889	0.099671	0.017761	0.070074	-0.03333	0.070813
	2	0.216993	0.107475	0.08773	0.06589	0.01881	0.062208	-0.02364	0.052104

Table 3.8 Bias and mean squared errors for $\hat{\lambda}_{B,1}$, T=350, $\sigma = 1, \lambda = 2$

	K	2		3		4		5	
		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	1.16952	1.55779	0.432404	0.326378	0.072819	0.13225	-0.14192	0.138203
	1	0.375411	0.388411	0.013462	2.98005	-0.1674	0.240543	-0.24407	0.28399
	1.5	0.282114	1.71339	-0.03202	12.5968	0.015992	0.743818	-0.04765	0.860576
	2	0.339766	20.2104	0.296583	17.8136	0.197537	17.7802	0.126943	25.8273

Table 3.9 Bias and mean squared errors for MLE, T=350, $\sigma = 1, \lambda = 2$

	K	2		3		4		5	
		MLE		MLE		MLE		MLE	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	-0.1397	0.887971	-0.25233	0.858669	-0.29672	0.866404	-0.28524	0.874577
	1	-0.13574	0.55183	-0.40445	0.536988	-0.49487	0.565588	-0.5101	0.572064
	1.5	0.121762	0.247277	-0.03015	0.254274	-0.06906	0.25701	-0.11586	0.265206
	2	0.112086	0.165208	0.038746	0.163288	0.030715	0.175685	0.003122	0.169451

Table 3.10 Bias and mean squared errors for $\hat{\lambda}_{B,1}$, T=350, $\sigma = 2, \lambda = 1$

K		2		3		4		5	
		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	1.675	2.8909	0.930504	0.924328	0.552422	0.347855	0.331015	0.147902
	1	0.640299	0.4788	0.259799	0.117755	0.071371	0.048742	-0.0322	0.045428
	1.5	0.349848	0.197353	0.081093	0.065058	-0.02616	0.058911	-0.09774	0.068515
	2	0.214874	0.356617	0.042835	0.79571	-0.04622	0.086509	-0.08557	0.087157

Table 3.11 Bias and mean squared errors for MLE, T=350, $\sigma = 2, \lambda = 1$

K		2		3		4		5	
		MLE		MLE		MLE		MLE	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	0.066974	0.494227	0.070702	0.480507	0.079445	0.471961	0.043866	0.471743
	1	0.054032	0.250681	-0.0678	0.166516	-0.13574	0.152342	-0.19382	0.153673
	1.5	0.159432	0.164778	-0.02055	0.112378	-0.103	0.102798	-0.14664	0.109465
	2	0.15504	0.112679	-0.02411	0.092666	-0.09112	0.091297	-0.11152	0.090875

Table 3.12 Bias and mean squared errors for $\hat{\lambda}_{B,1}$, T=350, $\sigma = 2, \lambda = 2$

K		2		3		4		5	
		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$		$\hat{\lambda}_{B,1}$	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	1.10754	1.38266	0.375249	0.253877	0.019931	0.103944	-0.20227	0.123791
	1	0.30659	0.28696	-0.06021	0.173617	-0.22162	0.213811	-0.2922	0.261571
	1.5	0.219925	4.26048	0.022397	0.372956	-0.07046	0.340778	-0.09007	0.334448
	2	0.18349	21.955	0.28368	4.80727	0.167541	11.4431	0.133622	11.7874

Table 3.13 Bias and mean squared errors for MLE, T=350, $\sigma = 2, \lambda = 2$

K		2		3		4		5	
		MLE		MLE		MLE		MLE	
		bias	mse	bias	mse	bias	mse	bias	mse
δ	0.5	-0.17824	0.971859	-0.14392	1.09994	-0.25619	1.08709	-0.27304	0.976421
	1	-0.38094	0.622765	-0.50632	0.617884	-0.5557	0.643139	-0.58182	0.638116
	1.5	-0.12991	0.344055	-0.24773	0.358703	-0.28482	0.370157	-0.27708	0.381657
	2	0.059415	0.180344	-0.01449	0.202585	-0.05958	0.198003	-0.08294	0.221809

Chapter Four

Inference Based On Upcrossings

4.1 Introduction:

Let N_j denote the number of upcrossings of u by $X(t)$ in $I_j = ((j-1)\delta, j\delta), \delta > 0$, $j=1, 2, \ldots, k$. Then $N_j \sim \mathcal{P}(c\lambda^{\frac{1}{2}})$, where $c = \frac{\delta}{2\pi\sigma} e^{-\frac{u^2}{2\sigma^2}}$. Also, $N_1, N_2, \ldots, N_k$ are asymptotically independent having the joint pdf

$$f(n_1, n_2, \ldots, n_k | \lambda, \sigma) = \prod_{j=1}^{k} e^{-c\lambda^{\frac{1}{2}}} \frac{(c\lambda^{\frac{1}{2}})^{n_j}}{n_j!},$$

$$= \frac{c^N \lambda^{\frac{N}{2}}}{\prod_{j=1}^{k} n_j!} e^{-ck\lambda^{\frac{1}{2}}},$$

when $N = n_1 + n_2 + \ldots + n_k$. It can be noted that $N \sim \mathcal{P}\left(kc\lambda^{\frac{1}{2}}\right)$.

In this chapter, we find several Bayes and classical inferences about λ based on the data $N_1, N_2, \ldots, N_k$. Two cases are considered namely σ^2 known and σ^2 is unknown. Also we consider different prior distributions for estimating σ^2 and λ.

4.2 The Maximum Likelihood Estimator Based on Upcrossings

4.2.1 Case 1. Known σ^2

Assuming σ^2 is known. Since $n_1, n_2, \ldots, n_k$ are iid $\mathcal{P}\left(c\lambda^{\frac{1}{2}}\right)$, then the MLE of $c\lambda^{\frac{1}{2}}$ is $\frac{N}{k}$. By the invariance property of MLE, the MLE of λ is

$$\hat{\lambda}_{MLE,2} = \left(\frac{N}{ck}\right)^2.$$

It is straight forward to check that

$$E(\hat{\lambda}_{MLE,2}) = E\left(\frac{N}{ck}\right)^2 = \frac{k^2c^2\lambda + kc\lambda^{\frac{1}{2}}}{k^2c^2},$$

$$= \lambda + \frac{\lambda^{\frac{1}{2}}}{kc}.$$

and

The bias of $\hat{\lambda}_{MLE,2} = \frac{\lambda^{\frac{1}{2}}}{kc} \to 0$ as $k \to \infty$. Similarly, the variance of $\hat{\lambda}_{MLE,2}$ is

$$Var(\hat{\lambda}_{MLE,2}) = \frac{1}{k^4c^4} Var(N^2) = \frac{1}{k^4c^4}\left(E(N^4) - \left(E(N^2)\right)^2\right),$$

$$= \frac{6\lambda^{\frac{3}{2}}}{kc} + \frac{7\lambda}{k^2c^2} + \frac{\lambda^{\frac{1}{2}}}{k^3c^3} - \frac{\lambda^{\frac{1}{4}}}{k^2c^2} \to 0 \text{ as } k \to \infty.$$

Since N is a complete sufficient statistics for λ and $E\left(\frac{N(N-1)}{k^2c^2}\right) = \lambda$, then

$\frac{N(N-1)}{k^2c^2}$ is a MVUE for λ.

Similarly, since $E\left(\frac{N}{kc}\right) = \lambda^{\frac{1}{2}}$, then

$\frac{N}{kc}$ is a MVUE for $\lambda^{\frac{1}{2}}$.

$$Var\left(\frac{N(N-1)}{k^2c^2}\right)=\frac{1}{k^4c^4}Var(N(N-1)),$$

$$=\frac{1}{k^4c^4}(E(N^2(N-1)^2)-(E(N(N-1))^2),$$

$$=\left(\frac{4\lambda^{\frac{3}{2}}}{kc}+\frac{2\lambda}{k^2c^2}\right)\to 0 \text{ as } k\to\infty.$$

$$\text{eff }(\hat{\lambda}_{MLE,2},\hat{\lambda}_{MVUE})=\frac{MSE(\hat{\lambda}_{MLE,2})}{MSE(\hat{\lambda}_{MVUE})},$$

$$=\frac{\frac{6\lambda^{\frac{3}{2}}}{kc}+\frac{7\lambda}{k^2c^2}+\frac{\lambda^{\frac{1}{2}}}{k^3c^3}-\frac{\lambda^{\frac{1}{4}}}{k^2c^2}+\frac{\lambda}{k^2c^2}}{\frac{4\lambda^{\frac{3}{2}}}{kc}+\frac{2\lambda}{k^2c^2}},$$

$$=\frac{6k^2c^2\lambda^{\frac{3}{2}}+\lambda^{\frac{1}{2}}+8kc\lambda-kc\lambda^{\frac{1}{4}}}{4k^2c^2\lambda^{\frac{3}{2}}+2kc\lambda}\to 1.5, \text{as } k\to\infty.$$

4.2.2 Case 2. σ^2 is unknown

Assuming that parameters $\sigma^2$and λ are unknown, then the MLE's of $\sigma^2$and λ are obtained as follows.

$$f(n_1,n_2,\ldots,n_k|\lambda,\sigma)=\frac{c_2^N\lambda^{\frac{N}{2}}(\sigma^2)^{-\frac{N}{2}}e^{-\frac{Nu^2}{2\sigma^2}}}{\prod_{j=1}^{k}n_j!}\ \exp\left(-c_2k\,\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}}\right)$$

$$\log f(n_1,n_2,\ldots,n_k|\lambda,\sigma^2)=\text{constant}+\frac{N}{2}\log\lambda-\frac{N}{2}\log\sigma^2-\frac{Nu^2}{2\sigma^2}-c_2k\,\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}}$$

Taking the first derivatives with respect to $\sigma^2$and λ yields,

$$\frac{d}{d\lambda}\log f(n_1, n_2, \dots, n_k|\lambda, \sigma^2) = \frac{N}{2\lambda} - \frac{c_2k\,(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}}}{2\lambda^{\frac{1}{2}}} = 0$$

and

$$\frac{d}{d\sigma^2}\log f(n_1, n_2, \dots, n_k|\lambda, \sigma^2) = -\frac{N}{2\sigma^2} + \frac{Nu^2}{2(\sigma^2)^2} + \frac{c_2k\,\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{3}{2}}e^{-\frac{u^2}{2\sigma^2}}}{2} - \frac{c_2k\,u^2\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{5}{2}}e^{-\frac{u^2}{2\sigma^2}}}{2} = 0.$$

Setting these derivatives to zero and simplifying them we get

$$\frac{N}{2\lambda} - \frac{c_2k\,(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}}}{2\lambda^{\frac{1}{2}}} = 0$$

$$-\frac{N}{2\sigma^2} + \frac{Nu^2}{2(\sigma^2)^2} + \frac{c_2k\,\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{3}{2}}e^{-\frac{u^2}{2\sigma^2}}}{2} - \frac{c_2k\,u^2\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{5}{2}}e^{-\frac{u^2}{2\sigma^2}}}{2} = 0$$

Solving these equations yields the MLE's of λ and σ^2 which are

$$\hat{\lambda}_{MLE,3} = \frac{N^2}{\left(c_2k\,(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}}\right)^2}.$$

$$\widehat{\sigma^2}_{MLE,3} = 0.$$

$$E(\hat{\lambda}_{MLE,3}) = \frac{ck\,\lambda^{\frac{1}{2}} + (ck\,\lambda^{\frac{1}{2}})^2}{\left(c_2 k\, e^{-\frac{u^2}{2\sigma^2}}\right)^2} = \left(\frac{c\lambda^{\frac{1}{2}}}{c_2 k} + \frac{c^2}{c_2^2}\lambda\right) e^{-u^2} \to \frac{c^2}{c_2^2}\lambda e^{-u^2} \text{ as } k \to \infty.$$

It is a biased estimator.

4.3 Bayes Estimation for λ when σ^2known

4.3.1 Jeffery's Prior

The Jeffery's prior is defined by $\pi_J(\lambda) \propto \sqrt{|I(\lambda)|}$, where $I(\lambda)$ is the Fisher information about λ contained in the sample. Since $N_1, N_2, \dots, N_k$ are iid, the Jeffery's prior of λ is derived as follows

$$I(\lambda) = -E\left(\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\log f(N_1|\lambda)\right) = \frac{c}{4}\lambda^{-\frac{3}{8}}.$$

Hence

$$\pi_J(\lambda) \propto \lambda^{-\frac{3}{8}}.$$

Under $\pi_J(\lambda)$ the posterior distribution and is

$$\pi_J(\lambda|n_1, n_2, \dots, n_k) \propto \lambda^{\frac{N}{2}-\frac{3}{8}} e^{-ck\lambda^{\frac{1}{2}}},$$

$$= c_3 \lambda^{\frac{N}{2}-\frac{3}{8}} e^{-ck\lambda^{\frac{1}{2}}},$$

where

$$c_3 = \left(\int_0^{\infty} \lambda^{\frac{N}{2}-\frac{3}{8}} e^{-ck\lambda^{\frac{1}{2}}}\, d\lambda\right)^{-1} = \frac{(ck)^{N+\frac{5}{4}}}{2\Gamma\left(N+\frac{5}{4}\right)}.$$

Finally,

$$\pi_J(\lambda|n_1, n_2, \ldots, n_k) = \frac{(ck)^{N+\frac{5}{4}}}{2\Gamma\left(N+\frac{5}{4}\right)} \lambda^{\frac{N}{2}-\frac{3}{8}} e^{-ck\lambda^{\frac{1}{2}}}, \qquad \lambda > 0.$$

Under Jeffery's prior, the Bayes estimator of λ is

$$\hat{\lambda}_b^J = E(\lambda|n_1, n_2, \ldots, n_k) = c_3 \int_0^{\infty} \lambda^{\frac{N}{2}+\frac{5}{8}} e^{-ck\lambda^{\frac{1}{2}}}\, d\lambda,$$

$$= \frac{(ck)^{N+\frac{5}{4}}}{2\Gamma\left(N+\frac{5}{4}\right)} \frac{2\Gamma\left(N+\frac{13}{4}\right)}{(ck)^{N+\frac{13}{4}}},$$

$$= (ck)^{-2}\left(N+\frac{9}{4}\right)\left(N+\frac{5}{4}\right).$$

$$E\left(\hat{\lambda}_b^J\right) = (ck)^{-2} E\left(N+\frac{9}{4}\right)\left(N+\frac{5}{4}\right) = \lambda + \frac{9\lambda^{\frac{1}{2}}}{2ck} + \frac{45}{16k^2c^2}.$$

Bias $\left(\hat{\lambda}_b^J\right) = \frac{9\lambda^{\frac{1}{2}}}{2ck} + \frac{45}{16k^2c^2} \to 0$ as $k \to \infty$.

It is possible to find a closed form for the mean square error of $\hat{\lambda}_b^J$. Since this mean square error is a function of noncentral moments of $\mathcal{P}(ck\lambda^{\frac{1}{2}})$ up to fourth order, then we will skip to use simulation to compare $\hat{\lambda}_b^J$ to it is counter parts.

To find the posterior variance, we find

$$E\left(\lambda^2 | n_1, n_2, \dots, n_k\right) = c_3 \int_0^{\infty} \lambda^{\frac{2N+5}{4}} e^{-ck\lambda^{\frac{1}{2}}} d\lambda,$$

$$= \frac{(ck)^{N+\frac{5}{4}}}{2\Gamma\left(N+\frac{5}{4}\right)} \frac{2\Gamma\left(N+\frac{21}{4}\right)}{(ck)^{N+\frac{21}{4}}},$$

$$= (ck)^{-4}\left(N+\frac{17}{4}\right)\left(N+\frac{13}{4}\right)\left(N+\frac{9}{4}\right)\left(N+\frac{5}{4}\right).$$

$$Var_{\pi}\left(\hat{\lambda}_b^J\right) = E(\lambda^2 | n_1, n_2, \dots, n_k) - \left(E\hat{\lambda}_b^J\right)^2,$$

$$= (ck)^{-4}\left(N+\frac{17}{4}\right)\left(N+\frac{13}{4}\right)\left(N+\frac{9}{4}\right)\left(N+\frac{5}{4}\right) -$$

$$(ck)^{-4}\left(N+\frac{9}{4}\right)^2\left(N+\frac{5}{4}\right)^2 \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Assuming σ is known, we find the predictive density of N_{k+1} given $n_1, n_2, \dots, n_{k+1}$. Hence

$$h(n_{k+1} | n_1, n_2, \dots, n_k, \sigma) = \int_0^{\infty} f(n_{k+1} | \lambda)\, \pi(\lambda | n_1, n_2, \dots, n_k) d\lambda$$

$$= \frac{c^{n_{k+1}}(ck)^{N+\frac{5}{4}}}{2n_{k+1}!\,\Gamma\left(N+\frac{5}{4}\right)} \int_0^\infty \lambda^{\frac{4N+4n_{k+1}-3}{8}} e^{-(c+ck)\lambda^{\frac{1}{2}}} d\lambda,$$

$$= \frac{c^{n_{k+1}}(ck)^{N+\frac{5}{4}}\Gamma\left(N+n_{k+1}+\frac{5}{4}\right)}{n_{k+1}!\,\Gamma\left(N+\frac{5}{4}\right)(c+ck)^{N+n_{k+1}+\frac{5}{4}}},$$

$$= \frac{\left(N+\frac{5}{4}\right)_{n_{k+1}}}{n_{k+1}!}\left(1-\frac{k}{k+1}\right)^{n_{k+1}}\left(\frac{k}{k+1}\right)^{N+\frac{5}{4}}, \quad n_{k+1}=0,1,2,\ldots$$

where

$$(a)_x = \frac{\Gamma(a+x)}{\Gamma(a)}.$$

According to Ghitany et al. (2001), the noncentral r^{th} moment of this distribution is

$$E(N_{k+1}^r | n_1, n_2, \ldots, n_k, \sigma) = \sum_{j=0}^{r} S(r,j) \frac{\left(N+\frac{5}{4}\right)_j}{\left(\frac{k}{k+1}\right)^j},$$

where $S(r,j)$ is the stirling's number of the second kind defined via

$$S(r,j) = \frac{1}{j!}\sum_{i=0}^{j}(-1)^i (j-i)^r.$$

Hence

$$E(N_{k+1} | n_1, n_2, \ldots, n_k, \sigma) = S(1,0)\left(N+\frac{5}{4}\right)_0 + \left(\frac{k+1}{k}\right) S(1,1)\left(N+\frac{5}{4}\right)_1$$

$$= \left(\frac{k+1}{k}\right)\left(N+\frac{5}{4}\right) \to N+\frac{1}{2}, \text{as } k \to \infty.$$

Similarly,

$$E(N_{k+1}^2|n_1,n_2,\dots,n_k,\sigma) = S(2,0)\frac{\left(N+\frac{5}{4}\right)_0}{\left(\frac{k}{k+1}\right)^0} + S(2,1)\frac{\left(N+\frac{5}{4}\right)_1}{\left(\frac{k}{k+1}\right)^1} + S(2,2)\frac{\left(N+\frac{5}{4}\right)_2}{\left(\frac{k}{k+1}\right)^2},$$

$$= \left(\frac{k+1}{k}\right)\left(N+\frac{5}{4}\right) + \frac{3}{2}\left(\frac{k+1}{k}\right)^2\left(N+\frac{9}{4}\right)\left(N+\frac{5}{4}\right),$$

The predictive variance is

$$Var(N_{k+1}|n_1,n_2,\dots,n_k,\sigma) = \left(\frac{k+1}{k}\right)\left(N+\frac{5}{4}\right) + \frac{3}{2}\left(\frac{k+1}{k}\right)^2\left(N+\frac{9}{4}\right)\left(N+\frac{5}{4}\right) - \left(\frac{k+1}{k}\right)^2\left(N+\frac{5}{4}\right)^2.$$

The variance of the errors of prediction is

$$Var(P.E) = Var\left(\left(\frac{k+1}{k}\right)\left(N+\frac{5}{4}\right) - N_{k+1}\right),$$

$$= Var\left(\left(\frac{k+1}{k}\right)\left(N+\frac{5}{4}\right) - N_{k+1}\right),$$

$$= \left(\frac{k+1}{k}\right)^2 Var\left(N+\frac{5}{4}\right) + VarN_{k+1},$$

$$= \left(\frac{k+1}{k}\right)^2 ck\lambda^{\frac{1}{2}} + c\lambda^{\frac{1}{2}},$$

$$= c\lambda^{\frac{1}{2}}\left(\frac{(k+1)^2}{k} + 1\right) \to 0$$

4.3.2 Estimation under Weibull prior

In this section, we find a Bayes estimators for the parameter λ under two cases namely σ^2 is known and σ^2 is unknown.

4.3.2.1 Case1. Known σ^2

To simplify the work, we may consider the Weibull distribution with parameters $\frac{1}{2}$ and b which a conjugate prior distribution for the parameter λ, i.e.,

$$\pi(\lambda) = \frac{1}{2b}\left(\frac{\lambda}{b}\right)^{-\frac{1}{2}} \exp\left(-\left(\frac{\lambda}{b}\right)^{\frac{1}{2}}\right), \qquad \lambda > 0.$$

Under this prior, the posterior distribution for λ is

$$\pi(\lambda|n_1, n_2, \dots, n_k, \sigma) \propto \lambda^{\frac{N}{2}}\lambda^{-\frac{1}{2}}\exp\left(-ck\lambda^{\frac{1}{2}}\right)\exp\left(-\left(\frac{\lambda}{b}\right)^{\frac{1}{2}}\right).$$

$$\propto \lambda^{\frac{N-1}{2}}\exp\left(-(ck + b^{-\frac{1}{2}})\lambda^{\frac{1}{2}}\right),$$

Hence,

$$\pi(\lambda|n_1, n_2, \dots, n_k, \sigma) = c_4\lambda^{\frac{N-1}{2}}\exp\left(-(ck + b^{-\frac{1}{2}})\lambda^{\frac{1}{2}}\right),$$

where

$$c_4 = \left(\int_0^\infty \lambda^{\frac{N-1}{2}}\exp\left(-(ck + b^{-\frac{1}{2}})\lambda^{\frac{1}{2}}\right)d\lambda\right)^{-1},$$

Using the transformation $v = \lambda^{\frac{1}{2}}$, c_4 reduces to

$$c_4 = \frac{\left(ck + b^{-\frac{1}{2}}\right)^{N+1}}{2\Gamma(N+1)}.$$

Finally, $\pi(\lambda|n_1, n_2, \dots, n_k, \sigma)$ takes the form

$$\pi(\lambda|n_1, n_2, \dots, n_k, \sigma) = \frac{\left(ck + b^{-\frac{1}{2}}\right)^{N+1}}{2\Gamma(N+1)} \lambda^{\frac{N-1}{2}} \exp\left(-(ck + b^{-\frac{1}{2}})\lambda^{\frac{1}{2}}\right), \quad \lambda > 0.$$

The Bayes estimator of λ under Weibull prior is $\hat{\lambda}_b^W$

$$\hat{\lambda}_b^W = c_4 \int_0^{\infty} \lambda^{\frac{N+1}{2}} \exp\left(-(ck + b^{-\frac{1}{2}})\lambda^{\frac{1}{2}}\right) d\lambda,$$

$$= c_4 \frac{2\Gamma(N+3)}{\left(ck + b^{-\frac{1}{2}}\right)^{N+3}} = \frac{\left(ck + b^{-\frac{1}{2}}\right)^{N+1} 2\Gamma(N+3)}{2\Gamma(N+1)\left(ck + b^{-\frac{1}{2}}\right)^{N+3}},$$

$$= \frac{(N+2)(N+1)}{\left(ck + b^{-\frac{1}{2}}\right)^{2}}.$$

Since, $\frac{N}{k} \xrightarrow{p} c\lambda^{\frac{1}{2}}$, then

$$\hat{\lambda}_b^W = \frac{\left(\frac{N}{k} + \frac{2}{k}\right)\left(\frac{N}{k} + \frac{1}{k}\right)}{\left(ck + b^{-\frac{1}{2}}\right)^{2}} \xrightarrow{p} \frac{c\lambda^{\frac{1}{2}} . c\lambda^{\frac{1}{2}}}{c^2} = \lambda, \text{ as } k \to \infty.$$

Also, the posterior variance of $\hat{\lambda}_b^W$ is calculated of follows

$$E\left(\lambda^2 | n_1, n_2, \dots, n_k\right) = c_4 \int_0^\infty \lambda^{\frac{N+3}{2}} \exp\left(-(ck + b^{-\frac{1}{2}})\lambda^{\frac{1}{2}}\right) d\lambda,$$

$$= c_4 \frac{2\Gamma(N+5)}{\left(ck + b^{-\frac{1}{2}}\right)^{N+5}} = \frac{\left(ck + b^{-\frac{1}{2}}\right)^{N+1} 2\Gamma(N+5)}{2\Gamma(N+1)\left(ck + b^{-\frac{1}{2}}\right)^{N+5}},$$

$$= \frac{(N+4)(N+3)(N+2)(N+1)}{\left(ck + b^{-\frac{1}{2}}\right)^4} \longrightarrow 0 \text{ as } k \to \infty.$$

$$Var\left(\hat{\lambda}_b^w\right) = E(\lambda^2 | n_1, n_2, \dots, n_k) - \left(E\hat{\lambda}_b^w\right)^2,$$

$$= \frac{(N+4)(N+3)(N+2)(N+1)}{\left(ck + b^{-\frac{1}{2}}\right)^4} - \frac{(N+2)^2(N+1)^2}{\left(ck + b^{-\frac{1}{2}}\right)^4}.$$

$$\longrightarrow 0 \text{ as } k \to \infty.$$

Assuming σ is known, we find the predictive density of N_{k+1} given $n_1, n_2, \dots, n_k$.

Hence

$$h(n_{k+1} | n_1, n_2, \dots, n_k, \sigma) = \int_0^\infty f(n_{k+1} | \lambda)\, \pi(\lambda | n_1, n_2, \dots, n_k, \sigma^2) d\lambda$$

$$= \frac{c^{n_{k+1}} \left(ck + b^{-\frac{1}{2}}\right)^{N+1}}{2 n_{k+1}!\, \Gamma(N+1)} \int_0^\infty \lambda^{\frac{N+n_{k+1}-1}{2}} e^{-(c+ck+b^{-\frac{1}{2}})\lambda^{\frac{1}{2}}} d\lambda,$$

$$= \frac{c^{n_{k+1}} \left(ck + b^{-\frac{1}{2}}\right)^{N+1} \Gamma(N + n_{k+1} + 1)}{n_{k+1}! \, \Gamma(N + 1) \left(c + ck + b^{-\frac{1}{2}}\right)^{N+n_{k+1}+1}},$$

$$= \frac{(N + 1)_{n_{k+1}}}{n_{k+1}!} \left(\frac{c}{c + ck + b^{-\frac{1}{2}}}\right)^{n_{k+1}} \left(\frac{ck + b^{-\frac{1}{2}}}{c + ck + b^{-\frac{1}{2}}}\right)^{N+1}, \quad n_{k+1} = 0,1,2, \ldots$$

It is mean and variance

$$E(N_{k+1} | n_1, n_2, \ldots, n_k, \sigma) = S(1,0)(N + 1)_0 + \left(\frac{c + ck + b^{-\frac{1}{2}}}{ck + b^{-\frac{1}{2}}}\right) S(1,1)(N + 1)_1$$

$$= \left(\frac{c + ck + b^{-\frac{1}{2}}}{ck + b^{-\frac{1}{2}}}\right)(N + 1).$$

Similarly,

$$E(N_{k+1}^2 | n_1, n_2, \ldots, n_k, \sigma) = S(2,0)(N + 1)_0 + S(2,1)\left(\frac{c+ck+b^{-\frac{1}{2}}}{ck+b^{-\frac{1}{2}}}\right)(N + 1)_1 +$$

$$S(2,2)\left(\frac{c + ck + b^{-\frac{1}{2}}}{ck + b^{-\frac{1}{2}}}\right)^2 (N + 1)_2,$$

$$= \left(\frac{c + ck + b^{-\frac{1}{2}}}{ck + b^{-\frac{1}{2}}}\right)(N + 1) +$$

$$\frac{3}{2}\left(\frac{c + ck + b^{-\frac{1}{2}}}{ck + b^{-\frac{1}{2}}}\right)^2 (N + 2)(N + 1).$$

The predictive variance is

$$Var(N_{k+1}|n_1,n_2,\dots,n_k,\sigma) = \left(\frac{c+ck+b^{-\frac{1}{2}}}{ck+b^{-\frac{1}{2}}}\right)(N+1) +$$

$$\frac{3}{2}\left(\frac{c+ck+b^{-\frac{1}{2}}}{ck+b^{-\frac{1}{2}}}\right)^2 (N+2)(N+1)$$

$$-\left(\frac{c+ck+b^{-\frac{1}{2}}}{ck+b^{-\frac{1}{2}}}\right)^2 (N+1)^2.$$

The variance of the errors of prediction is

$$Var(P.E) = Var\left(\left(\frac{c+ck+b^{-\frac{1}{2}}}{ck+b^{-\frac{1}{2}}}\right)(N+1) - N_{k+1}\right),$$

$$= \left(\frac{c+ck+b^{-\frac{1}{2}}}{ck+b^{-\frac{1}{2}}}\right)^2 Var(N+1) + VarN_{k+1},$$

$$= \left(\frac{c+ck+b^{-\frac{1}{2}}}{ck+b^{-\frac{1}{2}}}\right)^2 ck\lambda^{\frac{1}{2}} + c\lambda^{\frac{1}{2}},$$

$$= c\lambda^{\frac{1}{2}}\left(k\left(\frac{c+ck+b^{-\frac{1}{2}}}{ck+b^{-\frac{1}{2}}}\right)^2 + 1\right).$$

$$\to 0, u \to \infty \text{ or } \delta \to \infty.$$

$$\to 0, \delta \to 0 \text{ or } k \to \infty.$$

It can be noted that

$$k\left(\frac{c+ck+b^{-\frac{1}{2}}}{ck+b^{-\frac{1}{2}}}\right)^2+1\leq\frac{(k+1)^2}{k}+1,$$

if and only if

$$k\left(c+ck+b^{-\frac{1}{2}}\right)\leq(k+1)\left(ck+b^{-\frac{1}{2}}\right),$$

if and only if

$$0\leq b^{-\frac{1}{2}}.$$

4.4 Bayes Estimation for λ and σ^2

In the section, we assume that both parameters λ and σ^2 are unknown. We find Bayesian estimators for λ and σ^2. Since λ and $\sigma^2$are unknown, then we need to rewrite the pdf $f(n_1,n_2,\dots,n_k|\lambda,\sigma)$ in the following form.

$$f(n_1,n_2,\dots,n_k|\lambda,\sigma)=\prod_{j=1}^{k}e^{-c\lambda^{\frac{1}{2}}}\frac{(c\lambda^{\frac{1}{2}})^{n_j}}{n_j!},$$

$$=\frac{c^N\lambda^{\frac{N}{2}}}{\prod_{j=1}^{k}n_j!}e^{-ck\lambda^{\frac{1}{2}}}.$$

To simplify, we introduce the following notation. Let $c_2=\frac{\delta}{2\pi}$. Hence

$$f(n_1,n_2,\dots,n_k|\lambda,\sigma)=\frac{c_2^N\lambda^{\frac{N}{2}}(\sigma^2)^{-\frac{N}{2}}e^{-\frac{Nu^2}{2\sigma^2}}}{\prod_{j=1}^{k}n_j!}\exp\left(-c_2k\,\lambda^{\frac{1}{2}}(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}}\right),$$

We have tried to derive a Jeffery's prior for the parameters (λ,σ^2), but we found that it has a very complicated form. Therefore, we skip to use another prior for the case when both $\sigma^2$and λ are unknown. To complete the mission, we consider the following joint prior distribution for the parameters λ and σ^2.

$$\pi(\lambda|\sigma^2) \propto \lambda^{\frac{a_1}{2}} \exp\left(-c_2 k\, \lambda^{\frac{1}{2}} (\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right), \quad \sigma^2 > 0, \lambda > 0.$$

$$\pi(\sigma^2) = \frac{1}{b^{a_2}\Gamma(a_2)(\sigma^2)^{a_2+1}} \exp\left(-\frac{1}{b\sigma^2}\right), \quad \sigma^2 > 0.$$

i.e., we assume that σ^2 has an inverse Gamma distribution with parameters $a_2$and b while λ given is σ^2 has a Gamma distribution with parameters $(a_1 + 2)$ and $\frac{1}{H}$.

Under this prior, the joint posterior distribution for λ and σ^2 is

$$\pi(\lambda, \sigma^2|n_1, n_2, \dots, n_k) \propto \lambda^{\frac{N+a_1}{2}} \exp\left(-H\lambda^{\frac{1}{2}}\right) \frac{\exp\left(-\frac{1}{w\sigma^2}\right)}{(\sigma^2)^{\frac{N}{2}+a_2+1}}, \sigma^2 > 0, \lambda > 0.$$

where

$$H = 2c_2 k\, (\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}$$

and

$$w = \left(\frac{1}{b} + \frac{Nu^2}{2}\right)^{-1}$$

and c_5 is a normalizing constant given by

$$c_5 = \left(\int_0^{\infty} \lambda^{\frac{N+a_1}{2}} \exp\left(-H\lambda^{\frac{1}{2}}\right) d\lambda\right)^{-1} = \frac{H^{N+a_1+2}}{2\Gamma(N+a_1+2)}.$$

Hence

$$\pi(\lambda, \sigma^2 | n_1, n_2, \dots, n_k) \propto \lambda^{\frac{N+a_1}{2}} \exp\left(-H\lambda^{\frac{1}{2}}\right) H^{N+a_1+2} \times$$

$$\frac{\exp\left(-\frac{1}{w\sigma^2}\right)}{(\sigma^2)^{\frac{N}{2}+a_2+1}\left(2c_2k\,(\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right)^{N+a_1+2}},$$

$$\pi(\lambda, \sigma^2 | n_1, n_2, \dots, n_k) \propto \lambda^{\frac{N+a_1}{2}} \exp\left(-H\lambda^{\frac{1}{2}}\right) H^{N+a_1+2} \frac{\exp\left(-\frac{1}{w_1\sigma^2}\right)}{(\sigma^2)^{a_2-\frac{a_1}{2}}},$$

$$b < \frac{2}{u^2(a_1+2)}.$$

where $w_1 = \left(\frac{1}{w} - \frac{u^2(N+a_1+2)}{2}\right)^{-1}$. Therefore, the joint posterior density can be written follows

$$\pi(\lambda, \sigma^2 | n_1, n_2, \dots, n_k) \propto \pi(\lambda | \sigma^2, n_1, n_2, \dots, n_k)\pi(\sigma^2 | n_1, n_2, \dots, n_k),$$

Since $\pi(\lambda, \sigma^2 | n_1, n_2, \dots, n_k)$, $\pi(\lambda | \sigma^2, n_1, n_2, \dots, n_k)$ and $\pi(\sigma^2 | n_1, n_2, \dots, n_k)$ are density functions, then the normalizing constant is 1. Thus

$$\pi(\lambda, \sigma^2 | n_1, n_2, \dots, n_k) = \pi(\lambda | \sigma^2, n_1, n_2, \dots, n_k)\pi(\sigma^2 | n_1, n_2, \dots, n_k),$$

where

$\pi(\sigma^2|n_1, n_2, \dots, n_k)$ is the pdf of inverse Gamma distribution with parameters $(a_2 - \frac{a_1}{2} - 1)$ and w_1 and $\pi(\lambda|\sigma^2, n_1, n_2, \dots, n_k)$ is the pdf of a random variable λ such that $\lambda^{\frac{1}{2}}$ has Gamma distribution with parameters $(N + a_1 + 2)$ and $\frac{1}{H}$.

The purpose of writing the joint posterior in the conditional form is to ease simulation form it. To simulate an observation from $\pi(\lambda, \sigma^2|n_1, n_2, \dots, n_k)$ we design the following algorithm

i. Simulate $\sigma^2_{(i)}$ from inverse Gamma with $(a_2 - \frac{a_1}{2} - 1)$ and w_1.

ii. Use $\sigma^2_{(i)}$ in step (i) to calculate H.

iii. Simulate $\lambda_{(i)}$ from Gamma with parameters $(N + a_1 + 2)$ and $\frac{1}{H}$. Then $(\lambda_{(i)}, \sigma^2_{(i)})$ is a realization from $\pi(\lambda, \sigma^2|n_1, n_2, \dots, n_k)$.

iv. Repeat the steps (i)-(iii) L times. The realizations (λ, σ^2), $i = 1, \dots, L$ repeat a large sample from the joint posterior distribution and can be used to find the required summaries about the parameters λ and σ^2 or any functions of them. For example, the Bayes estimator of λ and σ^2 are respectively approximated by

$$\hat{\lambda} \approx \frac{1}{L}\sum_{i=1}^{L} \lambda_{(i)}$$

and

$$\widehat{\sigma^2} \approx \frac{1}{L}\sum_{i=1}^{L} \sigma^2_{(i)}.$$

The posterior mean of λ is

$$E(\lambda|n_1, n_2, \dots, n_k) = E\big(E(\lambda|\sigma^2, n_1, n_2, \dots, n_k)\big),$$

where the outer expectation is taken over σ^2. To find the inner expectation, the conditioned posterior distribution of λ given $\sigma^2, n_1, n_2, \dots, n_k$ is required. Since

$\lambda^{\frac{1}{2}}|\sigma^2, n_1, n_2, \dots, n_k \sim$Gamma $(N + a_1 + 2), \frac{1}{H}$), then

$$E(\lambda|\sigma^2, n_1, n_2, \dots, n_k) = E\left((\lambda^{\frac{1}{2}})^2|\sigma^2, n_1, n_2, \dots, n_k\right)$$

$$= Var\left(\lambda^{\frac{1}{2}}|\sigma^2, n_1, n_2, \dots, n_k\right) + \left(E\left(\lambda^{\frac{1}{2}}|\sigma^2, n_1, n_2, \dots, n_k\right)\right)^2,$$

$$= (N + a_1 + 2) + \frac{(N + a_1 + 2)^2}{H^2}.$$

Now, we take the expectation of the last equation w.r.t σ^2:

$$E(\lambda|n_1, n_2, \dots, n_k) = E\big(E(\lambda|\sigma^2, n_1, n_2, \dots, n_k)\big)$$

$$= (N + a_1 + 2) + (N + a_1 + 2)^2 E\left(\frac{1}{H^2}\right),$$

$$= (N + a_1 + 2) + (N + a_1 + 2)^2 \times$$

$$\int_0^\infty \frac{\exp\left(-\frac{1}{w_1\sigma^2}\right) d\sigma^2}{\left(2c_2k\,(\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right)^2 (\sigma^2)^{a_2-\frac{a_1}{2}}},$$

$$= (N + a_1 + 2) + (N + a_1 + 2)^2 \times$$

$$(2c_2k)^{-2} \int_0^\infty \frac{\exp\left(-\left(\frac{1}{w} - u^2\right)\frac{1}{\sigma^2}\right) d\sigma^2}{(\sigma^2)^{a_2-\frac{a_1}{2}-1}},$$

$$= (N + a_1 + 2) + \frac{(N + a_1 + 2)^2 \Gamma\left(a_2 - \frac{a_1}{2} - 2\right)}{4(c_2k)^2 \left(\frac{1}{w} - u^2\right)^{a_2-\frac{a_1}{2}-2}}.$$

Now, we turn to derive the predictive density of N_{k+1} when both λ and σ^2 are unknown according above prior. So

$$h(n_{k+1}|n_1, n_2, \dots, n_k) = \int_0^\infty \int_0^\infty f(n_{k+1}|\lambda, \sigma^2)\, \pi(\lambda, \sigma^2|n_1, n_2, \dots, n_k) d\lambda d\sigma^2$$

$$f(n_{k+1}|\lambda, \sigma) = \frac{c_2^{n_{k+1}} \lambda^{\frac{n_{k+1}}{2}} (\sigma^2)^{-\frac{n_{k+1}}{2}} e^{-\frac{u^2 n_{k+1}}{2\sigma^2}}}{n_{k+1}!} \exp\left(-H\lambda^{\frac{1}{2}}\right)$$

$$\pi(\lambda,\sigma^2|n_1,n_2,\dots,n_k)=\lambda^{\frac{N+a_1}{2}}\exp\left(-H\lambda^{\frac{1}{2}}\right)H^{N+a_1+2}\frac{\exp\left(-\frac{1}{w_1\sigma^2}\right)}{(\sigma^2)^{a_2-\frac{a_1}{2}}}$$

$$h(n_{k+1}|n_1,n_2,\dots,n_k)=\frac{c_2^{n_{k+1}}}{n_{k+1}!}H^{N+a_1+2}\times$$

$$\left(\int_0^\infty\frac{\exp\left(-\frac{1}{w_2\sigma^2}\right)}{(\sigma^2)^{\frac{n_{k+1}-a_1}{2}+a_2}}\left(\int_0^\infty\lambda^{\frac{N+n_{k+1}+a_1}{2}}\exp\left(-H_1\lambda^{\frac{1}{2}}\right)d\lambda\right)d\sigma^2\right),$$

where $w_2=\left(\frac{u^2n_{k+1}}{2}+\frac{1}{w_1}\right)^{-1}$,

and

$$H_1=3c_2k\,(\sigma^2)^{-\frac{1}{2}}e^{-\frac{u^2}{2\sigma^2}},$$

$$h(n_{k+1}|n_1,n_2,\dots,n_k)=\frac{2c_2^{n_{k+1}}\Gamma(N+n_{k+1}+a_1+2)H^{N+a_1+2}}{n_{k+1}!\,H_1^{N+n_{k+1}+a_1+2}}\times$$

$$\frac{\Gamma\left(\frac{n_{k+1}-a_1}{2}+a_2-1\right)}{w_2^{-\left(\frac{n_{k+1}-a_1}{2}+a_2-1\right)}}.$$

$$=\frac{(N+a_1+2)_{n_{k+1}}}{n_{k+1}!}\left(\frac{c_2}{H_1}\right)^{n_{k+1}}\left(\frac{H}{H_1}\right)^{N+a_1+2}\frac{\Gamma\left(\frac{n_{k+1}-a_1}{2}+a_2-1\right)}{w_2^{-\left(\frac{n_{k+1}-a_1}{2}+a_2-1\right)}}.$$

In the section, we assume that both parameters λ and σ^2 are unknown. We find Bayesian estimators for λ and σ^2. Since λ and σ^2 are unknown, then we need to rewrite the pdf $f(n_1, n_2, \ldots, n_k|\lambda, \sigma)$ in the following form.

$$f(n_1, n_2, \ldots, n_k|\lambda, \sigma) = \prod_{j=1}^{k} e^{-c\lambda^{\frac{1}{2}}} \frac{(c\lambda^{\frac{1}{2}})^{n_j}}{n_j!},$$

$$= \frac{c^N \lambda^{\frac{N}{2}}}{\prod_{j=1}^{k} n_j!} e^{-ck\lambda^{\frac{1}{2}}}.$$

To simplify, we introduce the following notation. Let $c_2 = \frac{\delta}{2\pi}$. Hence

$$f(n_1, n_2, \ldots, n_k|\lambda, \sigma) = \frac{c_2^N \lambda^{\frac{N}{2}} (\sigma^2)^{-\frac{N}{2}} e^{-\frac{Nu^2}{2\sigma^2}}}{\prod_{j=1}^{k} n_j!} \exp\left(-c_2 k\, \lambda^{\frac{1}{2}} (\sigma^2)^{-\frac{1}{2}} e^{-\frac{u^2}{2\sigma^2}}\right),$$

$$= \frac{c_2^N}{\prod_{j=1}^{k} n_j!} \sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j}{j!} \lambda^{\frac{N+j}{2}} (\sigma^2)^{-\left(\frac{N+j}{2}\right)} \exp\left(-\left(\frac{Nu^2 + ju^2}{2\sigma^2}\right)\right),$$

$$= \frac{c_2^N}{\prod_{j=1}^{k} n_j!} \sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j}{j!} \lambda^{\frac{N+j}{2}} (\sigma^2)^{-\left(\frac{N+j}{2}\right)} \exp\left(-\frac{u^2}{2}\left(\frac{N+j}{\sigma^2}\right)\right).$$

To complete this mission, we consider the following priors distribution for the parameters λ and σ^2.

$$\pi(\lambda) = \frac{1}{2b_1}\left(\frac{\lambda}{b_1}\right)^{-\frac{1}{2}} \exp\left(-\left(\frac{\lambda}{b_1}\right)^{\frac{1}{2}}\right), \quad \lambda > 0.$$

$$\pi(\sigma^2) = \frac{1}{b_2^a (\sigma^2)^{a+1}} \exp\left(-\left(\frac{1}{b_2\sigma^2}\right)\right), \quad \sigma^2 > 0.$$

i.e., we assume that σ^2 has an inverse Gamma distribution with parameters a and b_2 while λ has a Weibull distribution with parameters b_1 and $\frac{1}{2}$.

Under this prior, the joint posterior distribution for λ and σ^2 is

$$\pi(\lambda, \sigma^2 | n_1, n_2, \dots, n_k) \propto \sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j \lambda^{\frac{N+j-1}{2}}}{j!} \exp\left(-\frac{\lambda^{\frac{1}{2}}}{\sqrt{b_1}}\right) \times$$

$$\frac{\exp\left(-\left(\frac{u^2(N+j)}{2} + \frac{1}{b_2}\right)\frac{1}{\sigma^2}\right)}{(\sigma^2)^{\frac{N+j}{2}+a+1}},$$

$$\pi(\lambda, \sigma^2 | n_1, n_2, \dots, n_k) = c_3 \sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j \lambda^{\frac{N+j-1}{2}}}{j!} \exp\left(-\frac{\lambda^{\frac{1}{2}}}{\sqrt{b_1}}\right) \frac{\exp\left(-\frac{Q_j}{\sigma^2}\right)}{(\sigma^2)^{\frac{N+j}{2}+a+1}}.$$

where

$$Q_j = \frac{u^2(N+j)}{2} + \frac{1}{b_2} \quad , j = 0, 1, \dots$$

and

$$c_3 = \left(2 \sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j b_1^{\frac{N+j+1}{2}} \Gamma(N+j+1) \Gamma(\frac{N+j}{2} + a)}{j! \, Q_j^{\frac{N+j}{2}+a}}\right)^{-1},$$

The Bayes estimator of λ

$$\hat{\lambda} = c_3 \sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j}{j!} \left(\int_0^{\infty} \lambda^{\frac{N+j+1}{2}} \exp\left(-\frac{\lambda^{\frac{1}{2}}}{\sqrt{b_1}}\right) d\lambda \int_0^{\infty} \frac{\exp\left(-\frac{Q_j}{\sigma^2}\right)}{(\sigma^2)^{\frac{N+j}{2}+a+1}} d\sigma^2 \right),$$

$$\hat{\lambda} = \frac{\sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j}{j!} \frac{b_1^{\frac{N+j+3}{2}} \Gamma(N+j+3) \Gamma(\frac{N+j}{2}+a)}{Q_j^{\frac{N+j}{2}+a}}}{\sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j}{j!} \frac{b_1^{\frac{N+j+1}{2}} \Gamma(N+j+1) \Gamma(\frac{N+j}{2}+a)}{Q_j^{\frac{N+j}{2}+a}}}.$$

The Bayes estimator for σ^2 is

$$\widehat{\sigma^2} = c_3 \sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j}{j!} \left(\int_0^{\infty} \lambda^{\frac{N+j-1}{2}} \exp\left(-\frac{\lambda^{\frac{1}{2}}}{\sqrt{b_1}}\right) d\lambda \int_0^{\infty} \frac{\exp\left(-\frac{Q_j}{\sigma^2}\right)}{(\sigma^2)^{\frac{N+j}{2}+a}} d\sigma^2 \right),$$

$$\widehat{\sigma^2} = c_3 \sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j}{j!} \left(2 b_1^{\frac{N+j+1}{2}} \Gamma(N+j+1) \frac{\Gamma(\frac{N+j}{2}+a-1)}{Q_j^{\frac{N+j}{2}+a-1}} \right),$$

$$\widehat{\sigma^2} = \frac{\sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j}{j!} b_1^{\frac{N+j+1}{2}} \Gamma(N+j+1) \frac{\Gamma(\frac{N+j}{2}+a-1)}{Q_j^{\frac{N+j}{2}+a-1}}}{\sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j}{j!} \frac{b_1^{\frac{N+j+1}{2}} \Gamma(N+j+1) \Gamma(\frac{N+j}{2}+a)}{Q_j^{\frac{N+j}{2}+a}}}.$$

$$\pi(\lambda,\sigma^2|n_1,n_2,\dots,n_k) = c_3 \sum_{j=0}^{\infty} \frac{(c_2k)^j(-1)^j\lambda^{\frac{N+j-1}{2}}}{j!} \exp\left(-\frac{\lambda^{\frac{1}{2}}}{\sqrt{b_1}}\right) \frac{\exp\left(-\frac{Q_j}{\sigma^2}\right)}{(\sigma^2)^{\frac{N+j}{2}+a+1}},$$

$$= \sum_{j=1}^{\infty} w_j\, G\left(N+j+1, b_1^{\frac{1}{2}}\right) \times IG\left(\frac{N+j}{2}+a, Q_j\right),$$

where

$$w_j = \frac{\dfrac{(c_2k)^j(-1)^j\Gamma(N+j+1)\Gamma(\frac{N+j}{2}+a)b_1^{\frac{N+j+1}{2}}}{Q_j^{\frac{N+j}{2}+a}}}{\sum_{i=0}^{\infty}\dfrac{(c_2k)^i(-1)^i}{i!}\dfrac{\Gamma(N+i+1)\Gamma(\frac{N+i}{2}+a)b_1^{\frac{N+i+1}{2}}}{Q_i^{\frac{N+i}{2}+a}}}$$

Therefore, the joint posterior density can be written as follows

$\pi(\lambda,\sigma^2|n_1,n_2,\dots,n_k) = \sum_{j=1}^{\infty} w_j\, \pi_j(\lambda)\pi_j(\sigma^2).$

where

$\pi_j(\sigma^2)$ is the pdf of inverse Gamma distribution with parameters $\frac{N+j}{2}+a$ and Q_j

and $\pi_j(\lambda)$is the pdf of a random variable λ such that $\lambda^{\frac{1}{2}}$ has Gamma distribution

with parameters $N+j+1$and $b_1^{\frac{1}{2}}$.

To simulate an observation from $\pi(\lambda, \sigma^2 | n_1, n_2, \dots, n_k)$ we use the following algorithm

i. Simulate j from the distribution $p(J = j) = w_j, j = 0,1,2, \dots$

ii. Simulate $\lambda_{(i)}$ from $\pi_j(\lambda)$ and $\sigma^2_{(i)}$ from $\pi_j(\sigma^2)$.

iii. Then (λ, σ^2) is a realization from $\pi(\lambda, \sigma^2 | n_1, n_2, \dots, n_k)$.

v. Repeat the steps (i)-(iii) L times. The realizations $(\lambda_{(i)}, \sigma^2_{(i)})$, $i = 1, \dots, L$ represent a large sample from the joint posterior distribution and can be used to find the required summaries about the parameters λ and σ^2 or any functions of them. For example, the Bayes estimators of λ and σ^2 are respectively, approximated by

$$\hat{\lambda} \approx \frac{1}{L} \sum_{i=1}^{L} \lambda_{(i)}$$

and

$$\widehat{\sigma^2} \approx \frac{1}{L} \sum_{i=1}^{L} \sigma^2_{(i)}.$$

Now, we turn to derive the predictive density of N_{k+1} when both λ and σ^2 are unknown according to the above priors. So

The predictive density of N_{k+1} given $n_1, n_2, \dots, n_k$ is

$$h(n_{k+1}|n_1, n_2, \dots, n_k) = \int_0^\infty \int_0^\infty f(n_{k+1}|\lambda, \sigma^2)\, \pi(\lambda, \sigma^2|n_1, n_2, \dots, n_k) d\lambda d\sigma^2$$

where

$$f(n_{k+1}|\lambda, \sigma^2) = \frac{c_2^{n_{k+1}}}{n_{k+1}!} \sum_{j=0}^{\infty} \frac{{c_2}^j (-1)^j}{j!} \lambda^{\frac{n_{k+1}+j}{2}} (\sigma^2)^{-\left(\frac{n_{k+1}+j}{2}\right)} \exp\left(-\left(\frac{u^2 n_{k+1} + ju^2}{2\sigma^2}\right)\right),$$

and

$$\pi(\lambda, \sigma^2|n_1, n_2, \dots, n_k) = c_3 \sum_{j=0}^{\infty} \frac{(c_2 k)^j (-1)^j}{j!} \lambda^{\frac{N+j-1}{2}} \exp\left(-\frac{\lambda^{\frac{1}{2}}}{\sqrt{b_1}}\right) \frac{\exp\left(-\frac{Q_j}{\sigma^2}\right)}{(\sigma^2)^{\frac{N+j}{2}+a+1}}.$$

$$= \frac{c_3 c_2^{n_{k+1}}}{n_{k+1}!} \sum_{j=0}^{\infty} \frac{(c_2^2 k)^j (-1)^j}{j!^2} \lambda^{\frac{N+n_{k+1}+2j-1}{2}} \exp\left(-\frac{\lambda^{\frac{1}{2}}}{\sqrt{b_1}}\right) \times$$

$$\frac{\exp\left(-\frac{H_2}{\sigma^2}\right)}{(\sigma^2)^{\frac{N+n_{k+1}}{2}+j+a+1}},$$

where $H_2 = Q_j + \frac{u^2 n_{k+1} + ju^2}{2}$,

$$h(n_{k+1}|n_1, n_2, \dots, n_k) = \frac{c_3 c_2^{n_{k+1}}}{n_{k+1}!} \sum_{j=0}^{\infty} \frac{(c_2^2 k)^j (-1)^j}{j!^2} \times$$

$$\int_0^\infty \left(\frac{\exp\left(-\frac{H_2}{\sigma^2}\right)}{(\sigma^2)^{\frac{N+n_{k+1}}{2}+j+a+1}} \int_0^\infty \lambda^{\frac{N+n_{k+1}+2j-1}{2}} \exp\left(-\frac{\lambda^{\frac{1}{2}}}{\sqrt{b_1}}\right) d\lambda \right) d\sigma^2,$$

$$= \frac{c_3 c_2^{n_{k+1}}}{n_{k+1}!} \sum_{j=0}^{\infty} \frac{(c_2^2 k)^j (-1)^j}{j!^2} \times$$

$$\frac{2\Gamma(N + n_{k+1} + 2j + 1)\Gamma\left(\frac{N + n_{k+1}}{2} + j + a\right) b_1^{\frac{N+n_{k+1}+2j+1}{2}}}{H_2^{\frac{N+n_{k+1}}{2}+j+a}}.$$

4.5 Simulation

The aim of this section is to conduct a simulation study to the bias and MSE for the estimators of $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$. In this simulation the following different values of λ, σ, b and δ which are given Table 4.1.

Table 4.1 Values of $\sigma, \lambda, b,$ and δ used in simulation

σ	1 5
λ	2
δ	2 5 7
b	0.1 5 10

The results of the simulation are presented in tables (4.2)-(4.19). From Tables (4.2)-(4.19) we have to the following concluding remarks.

1. The |bias| and MSE's are decreasing functions in δ, for fixed b, k, σ and λ.
2. The |bias| and MSE's are decreasing as functions in k, for fixed b, σ, δ and λ.
3. We don't find any clear pattern in bias and MSE a functions in b for fixed δ, σ, δ and λ.

4. In term of bias and MSE, the estimator $\hat{\lambda}_{MLE,2}$ is the better than the estimator $\hat{\lambda}_b^J$.

5. In term of bias, the estimator $\hat{\lambda}_{MLE,2}$ is the better than the estimator $\hat{\lambda}_b^W$.

6. In term of bias and MSE, the $\hat{\lambda}_{MVUE}$ is the better than the estimator $\hat{\lambda}_b^J$.

7. In term of bias, the estimator $\hat{\lambda}_{MVUM}$is the best compared to classical estimators $\hat{\lambda}_b^J$ since it unbiased and has minimum variance over all unbiased estimators.

8. For some values of b, we may see that the $\hat{\lambda}_b^W$ is better than the other estimators in term of MSE. This due to the amount of information about λ contained in the prior.

Table 4.2 Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=0.1 , $\sigma = 1, \lambda = 2$

K		50							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	mse
δ	2	-0.78311	1.14387	0.841665	3.93047	0.26616	2.63085	-0.00755	2.25893
	5	-0.4156	0.626539	0.33798	1.15184	0.114226	0.950484	0.004597	0.889325
	7	-0.30192	0.486238	0.261383	0.794549	0.101798	0.685063	0.023152	0.650196

Table 4.3 Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=0.1 , $\sigma = 1, \lambda = 2$

K		100							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	mse
δ	2	-0.49143	0.739161	0.424319	1.54425	0.143139	1.22514	0.006061	1.12874
	5	-0.23699	0.358429	0.164402	0.494482	0.053792	0.446308	-0.00095	0.431637
	7	-0.17781	0.269554	0.114716	0.339713	0.035981	0.315701	-0.0031	0.30843

Table 4.4 Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=0.1 , $\sigma = 1, \lambda = 2$

K		150							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	mse
δ	2	-0.37239	0.565531	0.263323	0.929902	0.078385	0.796126	-0.01252	0.756005
	5	-0.15457	0.252425	0.121469	0.322104	0.047766	0.298939	0.011164	0.291507
	7	-0.11449	0.187082	0.08518	0.222736	0.0327	0.211119	0.006588	0.207404

Table 4.5 Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=5 , $\sigma = 1, \lambda = 2$

K		50							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	mse
δ	2	0.703449	3.11653	0.869915	4.04142	0.291938	2.69253	0.016995	2.29967
	5	0.281067	1.05167	0.324997	1.16721	0.101986	0.970043	-0.00727	0.910704
	7	0.186668	0.691104	0.214528	0.744014	0.056628	0.65135	-0.02118	0.624373

Table 4.6 Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=5 , $\sigma = 1, \lambda = 2$

K		100							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	Mse
δ	2	0.383304	1.35126	0.444524	1.54214	0.161866	1.21383	0.024048	1.1135
	5	0.142692	0.489537	0.160965	0.515714	0.050487	0.467424	-0.00419	0.452675
	7	0.096554	0.333859	0.108757	0.346352	0.030168	0.322888	-0.00884	0.315874

Table 4.7 Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=5 , $\sigma = 1, \lambda = 2$

K		150							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	Mse
δ	2	0.236894	0.840259	0.271489	0.916711	0.086096	0.781434	-0.00504	0.740685
	5	0.114035	0.322308	0.125649	0.334016	0.051892	0.309865	0.015263	0.30195
	7	0.072472	0.213404	0.080272	0.218837	0.027852	0.207807	0.001769	0.20438

Table 4.8 Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=10 , $\sigma = 1, \lambda = 2$

K = 50		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	bias	mse
δ	2	0.803408	3.43357	0.836676	3.87946	0.261471	2.59409	-0.01209	2.22864
	5	0.346557	1.15931	0.34562	1.21006	0.121641	1.00111	0.011899	0.936321
	7	0.235031	0.753465	0.231979	0.775616	0.073531	0.675558	-0.00455	0.64508

Table 4.9Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=10 , $\sigma = 1, \lambda = 2$

K = 100		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	bias	mse
δ	2	0.414334	1.51421	0.415955	1.60102	0.13554	1.2811	-0.00116	1.18403
	5	0.180885	0.518659	0.177835	0.528907	0.066925	0.476481	0.012032	0.459758
	7	0.109156	0.33197	0.106246	0.336516	0.027679	0.313832	-0.01132	0.307201

Table 4.10Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=10 , $\sigma = 1, \lambda = 2$

K = 150		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	mse
δ	2	0.283989	0.930608	0.28166	0.963471	0.096008	0.821058	0.004743	0.776896
	5	0.117509	0.307732	0.114828	0.311531	0.041231	0.289607	0.004683	0.28278
	7	0.071784	0.216142	0.069569	0.218066	0.01729	0.208126	-0.00872	0.20523

Table 4.11Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=0.1 , $\sigma = 5, \lambda = 2$

K		50							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	Mse
δ	2	-0.99132	1.49498	1.58542	10.1058	0.556528	5.67992	0.082579	4.42034
	5	-0.62829	0.917132	0.542733	2.23473	0.160263	1.66899	-0.02449	1.50505
	7	-0.4883	0.705881	0.398922	1.40595	0.126908	1.11856	-0.00579	1.03373

Table 4.12Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=0.1 , $\sigma = 5, \lambda = 2$

K		100							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	Mse
δ	2	-0.71215	1.06166	0.708379	3.19563	0.224689	2.25675	-0.00703	1.98163
	5	-0.34818	0.56319	0.310866	0.998647	0.120578	0.841796	0.027054	0.79185
	7	-0.2685	0.452931	0.214464	0.688566	0.079717	0.609955	0.013169	0.584753

Table 4.13Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=0.1 , $\sigma = 5, \lambda = 2$

K		150							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	Mse
δ	2	-0.53676	0.808461	0.482358	1.83297	0.162993	1.41877	0.007811	1.29383
	5	-0.2584	0.403038	0.193203	0.586257	0.067652	0.521931	0.005596	0.501954
	7	-0.18441	0.302139	0.147126	0.404566	0.057584	0.370127	0.01318	0.359062

Table 4.14Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=5 , $\sigma = 5, \lambda = 2$

	K	50							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	Mse
δ	2	1.15604	6.33318	1.54626	9.79508	0.524066	5.499	0.053464	4.29064
	5	0.532653	2.06124	0.629253	2.46553	0.239657	1.81794	0.051339	1.618
	7	0.332861	1.22699	0.389404	1.39257	0.117871	1.11181	-0.01459	1.03

Table 4.15Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=5 , $\sigma = 5, \lambda = 2$

	K	100							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	Mse
δ	2	0.601803	2.62058	0.731613	3.26879	0.245644	2.30129	0.012784	2.01403
	5	0.26449	0.885343	0.300821	0.968315	0.110892	0.817066	0.017547	0.769766
	7	0.179118	0.588168	0.202389	0.626865	0.067868	0.554629	0.001434	0.532516

Table 4.16Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=5 , $\sigma = 5, \lambda = 2$

	K	150							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	Mse
δ	2	0.433472	1.6337	0.506708	0.506708	0.185659	1.46269	0.029634	1.32916
	5	0.180439	0.554035	0.202131	0.58837	0.076296	0.52193	0.014098	0.500959
	7	0.10397	0.385442	0.117971	0.401697	0.029068	0.372348	-0.01502	0.363722

Table 4.17 Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=10 , $\sigma = 5, \lambda = 2$

K		50							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	Bias	Mse
δ	2	1.32774	7.43346	1.45739	9.3256	0.450524	5.26743	-0.01241	4.147
	5	0.541667	2.23068	0.55104	2.41525	0.168639	1.82328	-0.01608	1.64676
	7	0.384184	1.33431	0.384542	1.4065	0.113325	1.12738	-0.01898	1.04623

Table 4.18 Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=10 , $\sigma = 5, \lambda = 2$

K		100							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	bias	Mse
δ	2	0.713413	2.99011	0.735825	3.31348	0.249295	2.33963	0.016154	2.04958
	5	2.04958	0.913169	0.291172	0.945789	0.101583	0.799713	0.008409	0.754848
	7	0.210172	0.637258	0.207057	0.652869	0.072465	0.577944	0.005996	0.554525

Table 4.19 Bias and mean squared errors for $\hat{\lambda}_b^W, \hat{\lambda}_b^J, \hat{\lambda}_{MLE,2}$ and $\hat{\lambda}_{MVUE}$,b=10 , $\sigma = 5, \lambda = 2$

K		150							
		$\hat{\lambda}_b^W$		$\hat{\lambda}_b^J$		$\hat{\lambda}_{MLE,2}$		$\hat{\lambda}_{MVUE}$	
		Bias	mse	Bias	mse	Bias	mse	bias	Mse
δ	2	0.478626	1.73759	0.483217	1.85074	0.164016	1.4328	0.008915	1.30603
	5	0.192699	0.602165	0.189537	0.615861	0.064206	0.550675	0.00226	0.530243
	7	0.128987	0.390437	0.125936	0.396537	0.036848	0.366013	-0.00733	0.356829

Chapter Five

Application

5.1 Introduction

In these days, it is known that the wind is a good source of clean energy in power engineering. To generate a useful energy, it is known that most turbines need a wind speed between 7-10 mph. If the wind speed is very high, then the system should cease power generation to be protected from damage. On the other hand, a very low speed, i.e., less than 2mph can not be used to generate useful energy. Therefore, extreme values of the wind speed process are important since they can be used to assess the efficiency as well as the safety of a power generation system.

For example, the time that the wind speed process spends above a high threshold after an upcrossings could be used as a measure of system efficiency. In fact this time represents a duration of the wind speed process above a high threshold. Therefore, the distribution of this duration is required.

In this section, we are interested in applying the theory of this thesis to meteorological data. The data are the daily averages of wind speed at a station in the Republic of Ireland. The data consist of 6574 observed wind speed averages over the period 1961-1978 and are available online at *http:/lib.stat.cmu.edu/datasets*. The first step in analyzing this data is to transform the data to Gaussian process or near Gaussian process. If $X(t)$ denotes the wind speed average at time t, then the transformation $Y(t) = X(t)^{\frac{1}{2}}$ Gaussianizes the data. Also, we standardized the data so that $\sigma = 1$. Figure 5.1 shows the frequency histogram of the standardized data together with theoretical normal pdf. It seems to be that the data can be fitted by normal distribution. Also, a Kalmogorov-Smirnov test of normality is conduced and produced a p-value of 0.15, which means that the data provides us with no evidence to reject the normality. Figure 5.2 shows the plot of standardized wind speeds vs. time. The plot suggests a stationary process.

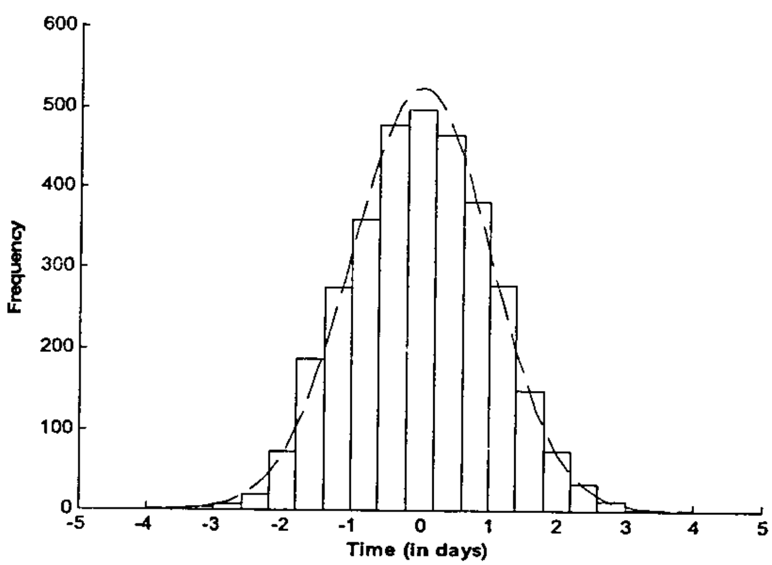

Figure 5.1 Histogram for wind speed data

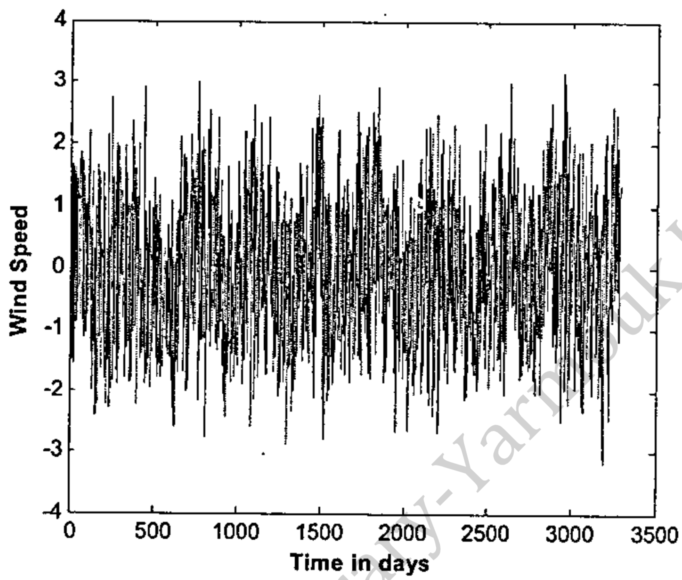

Figure 5.2 Time vs. Wind speed

The next steps in our analysis is to extract the durations and the number of upcrossings for wind speed process. To this end, we employ the matlab functions bwlabel in the image processing Toolbox. To test our theory, we divide the data into two parts. The first part represents the first 3287 observations and the remaining observations for the second part. The durations are extracted and classified into k=4 groups with δ=1.5. Using T=3287, we present these durations in Tables 5.1

Table 5.1.Durations extracted from wind data

Interval	Y_i(part I)	Y_i(part II)
[0, 1.5)	133	133
[1.5, 3)	29	28
[3, 4.5)	11	12
[4.5, ∞)	0	1

Also, the number of upcrossings for the first part of the data is $\sum_{j=1} N_j = 173$ and the number of upcrossings for second part is 174. The theory of this thesis produces the following estimators of λ

Table 5.2.Estimate and their standard errors

Estimator	Estimate	Standard error
$\hat{\lambda}_{B,1}$	1.15094	0.093239
$\hat{\lambda}_{MLE,1}$	2.44485	2.08758
$\hat{\lambda}_b^W$	1.0433	0.158411
$\hat{\lambda}_b^J$	1.04959	0.159595
$\hat{\lambda}_{MLE,2}$	1.03756	0.193788
$\hat{\lambda}_{MVUE}$	1.03157	0.157312

Tables 5.3.Predicted values of N_5

Predictor	Prediction	Prediction error
Under Jeffery's prior	173.579	124.495
Under Weibull prior	174.079	124.848

It can be noted that the predicted values are very close to the observed number of upcrossing 174, i.e., the theory produced an accurate prediction to the future number of upcrossing.

Chapter Six

Conclusions and Suggested Future Work

In this thesis, the probiem of estimating λ the variance of the derivative of a smooth stationary Gaussian process based on the durations and the crossings of high thresholds is tackled. Since the durations are observed up to intervals, grouped data approach was followed to find classical and Bayesian estimators. Also predictive distribution for a future duration derived. It is shown that the Bayes estimators as well as the predictive distribution and their characteristics have tractable close forms. The simulation study showed that the Bayesian estimators have some advantages over the classical estimators. On the other hand, we derived classical and Bayesian estimators for λ based on the number of upcrossings of high thresholds. Similarly, the simulation study also showed that the Bayes estimators based on upcrossings, have several advantage over the classical ones. Our simulation work is still not complete, since the simulation work of this thesis was restricted to the case of known σ. We believe that adding more simulated tables to the thesis for the case when σ is unknown is beyond the scope of this thesis. For this, we leave it to a future study. The hyper parameters r and b etc are assumed known in simulation. If they are unknown, then they will be replace by their estimates. In fact, this will add some uncertainty to the Bayes estimators. The effect of such uncertainty on the Bayes estimators could be considered for study.

As a future work, we propose the following tractable problems:

1. Deriving predictive distributions for

$$V_{min} = \min(V_1, \dots, V_M),$$

$$V_{max} = \max(V_1, \dots, V_M),$$

$$P = V_1 + \dots + V_M$$

and

$$P_i = \frac{V_i}{P}, \quad i = 1, 2, 3, \dots, M.$$

where M is the number of durations of $X(t)$ in a future interval of time, and $V_1, \dots, V_M$ are corresponding durations.

2. Deriving Bayesian intervals for λ such as high posterior density credible region (HPDCR).

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